Amit 問題(解答例)

第1章

第2章

2.1　2章の問題

1.
2.
3. (a)
   (b)
   (c)
   (d)
4.
5.

第3章

3.1 3章の問題

1.

2.

3.

4.

$$\begin{aligned}
T_{t_1t_2}\{J+\delta J\} &= T\left[\exp\left(i\int_{t_1}^{t_2}(J(x)+\delta J(x))\hat{\Phi}(x)d^4x\right)\right] \\
&= T\left[\exp\left(i\int_{t_1}^{t_2}J(x)\hat{\Phi}(x)d^4x + i\int_{t_1}^{t_2}\delta J(x)\hat{\Phi}(x)d^4x\right)\right] \\
&= T\left[\exp\left(i\int_{t_1}^{t_2}\delta J(x)\hat{\Phi}(x)d^4x\right)\exp\left(i\int_{t_1}^{t_2}J(x')\hat{\Phi}(x')d^4x'\right)\right] \\
&\approx T\left[\left(1+i\int_{t_1}^{t_2}\delta J(x)\hat{\Phi}(x)d^4x\right)\exp\left(i\int_{t_1}^{t_2}J(x')\hat{\Phi}(x')d^4x'\right)\right] \\
&= T\left[\exp\left(i\int_{t_1}^{t_2}J(x)\hat{\Phi}(x)d^4x\right)\right] + T\left[i\int_{t_1}^{t_2}\delta J(x)\hat{\Phi}(x)d^4x\exp\left(i\int_{t_1}^{t_2}J(x')\hat{\Phi}(x')d^4x'\right)\right] \\
&= T_{t_1t_2}\{J\} + \int_{t_1}^{t_2}\delta J(x)T\left[i\hat{\Phi}(x)T_{t_1t_2}\{J\}\right] \qquad (3.1)
\end{aligned}$$

2 行目から 3 行目の式変形には、時間順序積の関係 $T[\hat{A}\hat{B}] = T[\hat{B}\hat{A}]$ を使った (これは、同時刻で可換でない演算子があるときには成り立たない。この問題の場合は、$[\hat{\Phi}(x_0,\boldsymbol{x}),\hat{\Phi}(x_0,\boldsymbol{y})] = 0$ なので、$T[\hat{A}\hat{B}] = T[\hat{B}\hat{A}]$ で問題ない。)

5. まず、問題で与えられた式から

$$\begin{aligned}
&\int\mathcal{D}\phi\frac{\delta}{\delta\phi(y)}\left[\phi(x)\exp\left(-\int\mathcal{L}(\phi)dx'\right)\right] \\
&= \int\mathcal{D}\phi\delta(x-y)\exp\left(-\int\mathcal{L}(\phi)dx'\right) + \int\mathcal{D}\phi\phi(x)\frac{\delta}{\delta\phi(y)}\left[\exp\left(-\int\mathcal{L}(\phi)dx'\right)\right] = 0 \\
&(\because \delta\phi(x)/\delta\phi(y) = \delta(x-y)) \qquad (3.2)
\end{aligned}$$

ところで

$$\int\mathcal{D}\phi\phi(x)\frac{\delta}{\delta\phi(y)}\left[\exp\left(-\int\mathcal{L}dx'\right)\right] = \int\mathcal{D}\phi\phi(x)\left(-\frac{\delta}{\delta\phi(y)}\int\mathcal{L}dx'\right)\exp\left(-\int\mathcal{L}dx''\right) \qquad (3.3)$$

ここで、

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + P(\phi^2) \tag{3.4}$$

であることと、

$$\begin{aligned}
\frac{\delta}{\delta\phi(y)}\int(\partial\phi(x'))^2dx' &= 2\int\partial\phi(x')\frac{\delta(\partial\phi(x'))}{\delta\phi(y)}dx' = 2\int\partial\phi(x')\partial\left(\frac{\delta\phi(x')}{\delta\phi(y)}\right)dx' \\
&= 2\int\partial\left\{\partial\phi(x')\frac{\delta\phi(x')}{\delta\phi(y)}\right\}dx' - 2\int\partial^2\phi(x')\frac{\delta\phi(x')}{\delta\phi(y)}dx' \\
&= -2\int\partial^2\phi(y)\frac{\delta\phi(x')}{\delta\phi(y)}dx' = -2\int\partial^2\phi(y)\delta(x'-y)dx'
\end{aligned} \tag{3.5}$$

(表面積分の寄与を 0 とした)、および

$$\frac{\delta}{\delta\phi(y)}P(\phi^2(x')) = \frac{\delta(\phi^2(x'))}{\delta\phi(y)}\frac{\delta}{\delta(\phi^2(x'))}P(\phi^2(x')) = 2\phi(x)\delta(x'-y)P'(\phi^2(x')) \tag{3.6}$$

を使うと、

$$\begin{aligned}
&\int\mathcal{D}\phi\phi(x')\frac{\delta}{\delta\phi(y)}\left[\exp\left(-\int\mathcal{L}(\phi)dx'\right)\right] \\
&= \int\mathcal{D}\phi\phi(x)\left(\partial^2\phi(y) - \phi(y)P'(\phi^2(y))\right)\exp\left(-\int\mathcal{L}(\phi)dx''\right)
\end{aligned} \tag{3.7}$$

したがって、(3.2), (3.3),(3.7) より、

$$\partial_y^2\langle\phi(x)\phi(y)\rangle - \langle\phi(x)\phi(y)P'(\phi^2(y))\rangle = -\delta(x-y) \tag{3.8}$$

6.

7.

$$\begin{aligned}
&\partial_0\langle 0|T[\hat{\Phi}(x)T\{J\}]|0\rangle \\
&= \partial_0\langle 0|T[\hat{\Phi}(x)\exp\left(i\int J(x')\hat{\Phi}(x')d^4x'\right)]|0\rangle = \partial_0\langle 0|T[\hat{\Phi}(x)\exp\left(i\int_{-\infty}^{\infty}\hat{W}(x_0')x_0'\right)]|0\rangle \\
&= \partial_0\langle 0|T[\exp\left(i\int_{x_0}^{\infty}dx_0'\hat{W}(x_0')\right)\hat{\Phi}(x_0,\boldsymbol{x})\exp\left(i\int_{-\infty}^{x_0}dx_0''\hat{W}(x_0'')\right)]|0\rangle \\
&= \langle 0|T[\exp\left(i\int_{x_0}^{\infty}dx_0'\hat{W}(x_0')\right)\left\{-i\hat{W}(x_0) + \dot{\hat{\Phi}}(x_0,\boldsymbol{x}) + i\hat{W}(x_0)\right\}\exp\left(i\int_{-\infty}^{x_0}dx_0''\hat{W}(x_0'')\right)]|0\rangle \\
&= \langle 0|T[\exp\left(i\int_{x_0}^{\infty}dx_0'\hat{W}(x_0')\right)\dot{\hat{\Phi}}(x_0,\boldsymbol{x})\exp\left(i\int_{-\infty}^{x_0}dx_0''\hat{W}(x_0'')\right)]|0\rangle
\end{aligned} \tag{3.9}$$

(ここで

$$\hat{W}(y_0) \equiv \int d^3\boldsymbol{y}J(y)\hat{\Phi}(y) \tag{3.10}$$

と置いた。また関係する演算子は同時刻で交換する ($[\hat{\Phi}(x_0,\boldsymbol{x}),\hat{\Phi}(x_0,\boldsymbol{y})] = 0$) ので、時間順序積で $T[\hat{B}\hat{A}] = T[\hat{A}\hat{B}]$ を使うことができて通常の演算規則が使える。)

もう一回時間微分する時、同時刻で $\Phi(x_0,\boldsymbol{x}')$ と $\dot{\Phi}(x_0,\boldsymbol{x})$ が交換しないことに注意して、

$$
\begin{aligned}
&\partial_0^2\langle 0|T[\Phi(x)T\{J\}]0\rangle \\
&=\langle 0|T[\exp\left(i\int_{x_0}^{\infty}dx_0'\hat{W}(x_0')\right)\left\{-i\hat{W}(x_0)\dot{\hat{\Phi}}(x_0,\boldsymbol{x})+\ddot{\hat{\Phi}}(x_0,\boldsymbol{x})+\dot{\hat{\Phi}}(x_0,\boldsymbol{x})i\hat{W}(x_0)\right\}\exp\left(i\int_{-\infty}^{x_0}dx_0''\hat{W}(x_0'')\right)]|0\rangle \\
&=\langle 0|T[\exp\left(i\int_{x_0}^{\infty}dx_0'\hat{W}(x_0')\right)\left\{\ddot{\hat{\Phi}}(x_0,\boldsymbol{x})-i\int d^3\boldsymbol{x}'J(x_0,\boldsymbol{x}')[\hat{\Phi}(x_0,\boldsymbol{x}'),\dot{\hat{\Phi}}(x_0,\boldsymbol{x})]\right\}\exp\left(i\int_{-\infty}^{x_0}dx_0''\hat{W}(x_0'')\right)]|0\rangle \\
&=\langle 0|T[\left\{\ddot{\hat{\Phi}}(x_0,\boldsymbol{x})-i\int d^3\boldsymbol{x}'J(x_0,\boldsymbol{x}')[\hat{\Phi}(x_0,\boldsymbol{x}'),\dot{\hat{\Phi}}(x_0,\boldsymbol{x})]\right\}\exp\left(i\int_{-\infty}^{\infty}dx_0''\hat{W}(x_0'')\right)]|0\rangle \\
&=\langle 0|T[\partial_0^2\hat{\Phi}(x)T\{J\}|0\rangle-i\int d^3\boldsymbol{x}'J(x_0,\boldsymbol{x}')\langle 0|T[[\hat{\Phi}(x_0,\boldsymbol{x}'),\dot{\hat{\Phi}}(x_0,\boldsymbol{x})]T\{J\}]|0\rangle \qquad (3.11)
\end{aligned}
$$

8. まず、以下のことを示す。

$$\left[f\left(\frac{\delta}{\delta J(y)}\right),J(x)\right]=\delta(x-y)f'\left(\frac{\delta}{\delta J(y)}\right) \qquad (3.12)$$

(a)

$$\left[\frac{\delta}{\delta J(y)},J(x)\right]=\delta(x-y) \qquad (3.13)$$

(b)

$$\left[\left(\frac{\delta}{\delta J(y)}\right)^n,J(x)\right]=\delta(x-y)n\left(\frac{\delta}{\delta J(y)}\right)^{n-1} \qquad (3.14)$$

を仮定する。すると、

$$
\begin{aligned}
\left[\left(\frac{\delta}{\delta J(y)}\right)^{n+1},J(x)\right]&=\left[\left(\frac{\delta}{\delta J(y)}\right)\left(\frac{\delta}{\delta J(y)}\right)^{n},J(x)\right] \\
&=\left(\frac{\delta}{\delta J(y)}\right)\left[\left(\frac{\delta}{\delta J(y)}\right)^{n},J(x)\right]+\left[\left(\frac{\delta}{\delta J(y)}\right),J(x)\right]\left(\frac{\delta}{\delta J(y)}\right)^{n} \\
&=\left(\frac{\delta}{\delta J(y)}\right)\delta(x-y)n\left(\frac{\delta}{\delta J(y)}\right)^{n-1}+\delta(x-y)\left(\frac{\delta}{\delta J(y)}\right)^{n} \\
&=\delta(x-y)(n+1)\left(\frac{\delta}{\delta J(y)}\right)^{n} \qquad (3.15)
\end{aligned}
$$

(c) 数学的帰納法より、一般に

$$\left[\left(\frac{\delta}{\delta J(y)}\right)^n,J(x)\right]=\delta(x-y)n\left(\frac{\delta}{\delta J(y)}\right)^{n-1} \qquad (3.16)$$

が成立。

これから $f(x)$ が　Taylor 展開可能な関数ならば、(3.12) が成り立つ。

(3.12) を使うと、

$$\begin{aligned}
&\left[\exp\left\{i\int F\left(-i\frac{\delta}{\delta J(y)}\right)\right\}d^4y, J(x)\right] \\
&= \int \delta(x-y)F'\left(-i\frac{\delta}{\delta J(y)}\right)d^4y \exp\left\{i\int F\left(-i\frac{\delta}{\delta J(y')}\right)d^4y'\right\} \\
&= F'\left(-i\frac{\delta}{\delta J(x)}\right)\exp\left\{i\int F\left(-i\frac{\delta}{\delta J(y)}\right)d^4y\right\} \qquad (3.17)
\end{aligned}$$

9.

$$\begin{aligned}
[\hat{\Phi}^+(x), \hat{\Phi}^-(y)] &= i\Delta^+(x-y) \\
&= \frac{1}{(2\pi)^3}\int d^4k\delta(k^2+m^2)\theta(k_0)\exp(ik(x-y)) \qquad (3.18)
\end{aligned}$$

つまり、

$$\Delta^+(x) = \frac{-i}{(2\pi)^3}\int d^4k\delta(k^2+m^2)\theta(k_0)\exp(ikx) \qquad (3.19)$$

10.

$$\Delta_c(x-y) = -\theta(x_0-y_0)\Delta^+(x-y) - \theta(y_0-x_0)\Delta^+(y-x) \qquad (3.20)$$

が Klein-Gordon 方程式の Green 関数であることを示す。

(a) まず、

$$\begin{aligned}
\Delta^+(x_0, -\boldsymbol{x}) &= \frac{-i}{(2\pi)^3}\int d^4k\delta(k^2+m^2)\theta(k_0)\exp(-ik_0x_0)\exp(i\boldsymbol{k}\cdot(-\boldsymbol{x})) \\
&= \frac{-i}{(2\pi)^3}\int d^4k'\delta(k'^2+m^2)\theta(k'_0)\exp(-ik_0x_0)\exp(i(-\boldsymbol{k}')\cdot(-\boldsymbol{x})) \\
&= \Delta^+(x_0, \boldsymbol{x}) \qquad (3.21)
\end{aligned}$$

(1 行目から 2 行目で $k' = (k_0, -\boldsymbol{k})$ と変数変換した)

(b) 次に、微分演算子 $K_x \equiv m^2 - \partial^2$ について

$$\begin{aligned}
K_x\Delta^+(x) &= \frac{-i}{(2\pi)^3}K_x\int d^4k\delta(k^2+m^2)\theta(k_0)\exp(ikx) \\
&= \frac{i}{(2\pi)^3}\int d^4k(k^2+m^2)\delta(k^2+m^2)\theta(k_0)\exp(ikx) = 0 \qquad (3.22)
\end{aligned}$$

(ここで $y\delta(y) = 0$ を使った。)

(c)

$$\begin{aligned}
\partial_{x_0}\Delta^+(x) &= \frac{-i}{(2\pi)^3}\partial_{x_0}\int d^4k\delta(k^2+m^2)\theta(k_0)\exp(ikx) \\
&= \frac{-i}{(2\pi)^3}\int d^4k\delta(k^2+m^2)\theta(k_0)(-ik_0)\exp(ikx) \\
&= \frac{-1}{2(2\pi)^3}\int d^3\boldsymbol{x}\exp(-i\omega(\boldsymbol{k})x_0)\exp(i\boldsymbol{k}\cdot\boldsymbol{x}) \qquad (3.23)
\end{aligned}$$

(上式の 2 行目から 3 行目では $\delta(k^2+m^2)=\delta((\omega(\boldsymbol{k})+k_0)(\omega(\boldsymbol{k})-k_0))=\frac{1}{2\omega(\boldsymbol{k})}(\delta(\omega(\boldsymbol{k}+k_0)+\delta(\omega(\boldsymbol{k}-k_0));\ \omega(\boldsymbol{k})=\sqrt{m^2+\boldsymbol{k}^2}$ を使い、さらに k_0 について積分した。)
同様にして

$$\partial_{x_0}\Delta^+(-x)=\frac{1}{2(2\pi)^3}\int d^3\boldsymbol{x}\exp(i\omega(\boldsymbol{k})x_0)\exp(-i\boldsymbol{k}\cdot\boldsymbol{x}) \tag{3.24}$$

すると、

$$\begin{aligned}
&\partial_{x_0}\Delta_c(x)\\
&=-\delta(x_0)\Delta^+(x)-\theta(x_0)\partial_{x_0}\Delta^+(x)+\delta(-x_0)\Delta^+(-x)-\theta(-x_0)\partial_{x_0}\Delta^+(-x)\\
&=-\theta(x_0)\partial_{x_0}\Delta^+(x)-\theta(-x_0)\partial_{x_0}\Delta^+(-x)\\
&(\because \text{式 (3.21) から}\delta(x_0)(\Delta^+(x)-\Delta^+(-x))=0)
\end{aligned} \tag{3.25}$$

さらに、

$$\partial_{x_0}^2\Delta_c(x)=-\delta(x_0)\partial_{x_0}\Delta^+(x)-\theta(x_0)\partial_{x_0}^2\Delta^+(x)+\delta(-x_0)\partial_{x_0}\Delta^+(-x)-\theta(-x_0)\partial_{x_0}^2\Delta^+(-x) \tag{3.26}$$

従って、

$$\begin{aligned}
K_x\Delta_c(x)&=-\delta(x_0)\partial_{x_0}\Delta^+(x)-\theta(x_0)K_x\Delta^+(x)+\delta(-x_0)\partial_{x_0}\Delta^+(-x)-\theta(-x_0)K_x\Delta^+(-x)\\
&=-\delta(x_0)\partial_{x_0}\Delta^+(x)+\delta(-x_0)\partial_{x_0}\Delta^+(-x)\\
&=\delta(x_0)\delta^3(\boldsymbol{x})=\delta^4(x)
\end{aligned} \tag{3.27}$$

(ここで、1 行目から 2 行目では (3.22) を、2 行目から 3 行目では (3.23),(3.24) 使った。)

11. まず階段関数のフーリエ変換は

$$\theta(t)=\frac{i}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{\exp(-i\omega t)}{\omega+i\epsilon}\qquad(0<\epsilon\ll 1) \tag{3.28}$$

したがって

$$\begin{aligned}
\theta(x_0)\Delta^+(x)&=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}d\omega\frac{\exp(-i\omega x_0)}{\omega+i\epsilon}\int d^4k\delta(k^2+m^2)\theta(k_0)\exp(ikx)\\
&=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}d\omega d^3\boldsymbol{k}\frac{\exp(-i(\omega+\omega(\boldsymbol{k}))x_0)}{2\omega(\boldsymbol{k})(\omega+i\epsilon)}\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\\
&=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}d^4k\frac{\exp(-ik_0x_0)}{2\omega(\boldsymbol{k})(k_0-\omega(\boldsymbol{k})+i\epsilon)}\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\\
&=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}d^4k\frac{1}{2\omega(\boldsymbol{k})(k_0-\omega(\boldsymbol{k})+i\epsilon)}\exp(ikx)
\end{aligned} \tag{3.29}$$

上の式の 1 行目から 2 行目の式変形には $\delta(k^2+m^2)=\delta((\omega(\boldsymbol{k})+k_0)(\omega(\boldsymbol{k})-k_0))=\frac{1}{2\omega(\boldsymbol{k})}(\delta(\omega(\boldsymbol{k}+k_0)+\delta(\omega(\boldsymbol{k}-k_0));\ \omega(\boldsymbol{k})=\sqrt{m^2+\boldsymbol{k}^2}$ を使っている。また 2 行目から 3 行目では $k_0=\omega+\omega(\boldsymbol{k})$ と変数変換した。

同様にして、

$$\begin{aligned}\theta(-x_0)\Delta^+(-x) &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k \frac{1}{2\omega(\boldsymbol{k})(k_0-\omega(\boldsymbol{k})+i\epsilon)}\exp(ik(-x)) \\ &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k \frac{1}{2\omega(\boldsymbol{k})(-k_0-\omega(\boldsymbol{k})+i\epsilon)}\exp(ikx) \qquad (3.30)\end{aligned}$$

($k \to -k$ と変数変換し、$\omega(-\boldsymbol{k}) = \omega(\boldsymbol{k})$ を使った。) 従って、

$$\begin{aligned}\Delta_c(x) &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k \frac{1}{2\omega(\boldsymbol{k})}\left(\frac{1}{k_0-\omega(\boldsymbol{k})+i\epsilon}+\frac{1}{-k_0-\omega(\boldsymbol{k})+i\epsilon}\right)\exp(ikx) \\ &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k \frac{1}{\omega(\boldsymbol{k})^2-k_0^2-i\epsilon'}\exp(ikx) \qquad (0<\epsilon'\ll 1) \\ &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k \frac{1}{k^2+m^2-i\epsilon'}\exp(ikx) \qquad (3.31)\end{aligned}$$

12.

13.

$$\Delta_R(x) = -\theta(x_0)\Delta(x), \quad \Delta(x) = \Delta^+(x) - \Delta^+(-x) \qquad (3.32)$$

1 回時間微分して

$$\partial_0\Delta_R(x) = -\delta(x_0)\Delta(x) - \theta(x_0)\partial_0\Delta(x) = -\theta(x_0)\partial_0\Delta(x) \qquad (3.33)$$

(ここで (3.21) を使っている。) さらに時間微分すると

$$\partial_0^2\Delta_R(x) = -\delta(x_0)\partial_0\Delta(x) - \theta(x_0)\partial_0^2\Delta(x) \qquad (3.34)$$

$$K_x\Delta_R(x) = -\delta(x_0)\partial_0\Delta(x) - \theta(x_0)K_x\Delta(x) = -\delta(x_0)\partial_0\Delta(x) \qquad (3.35)$$

(ここで (3.22) 式を使った。) さらに (3.23) 式　を使うと、

$$K_x\Delta_R(x) = \delta(x_0)\delta^3(\boldsymbol{x}) = \delta^4(x) \qquad (3.36)$$

$\Delta_A(x) = \theta(-x_0)\Delta(x)$ (Amit の定義はミスプリ) についても同様な手順で $K_x\Delta_A(x) = \delta^4(x)$

14.

$$\hat{\Phi}_{IN}(x) = \hat{\Phi}^+_{IN}(x) + \hat{\Phi}^-_{IN}(x) \qquad (3.37)$$

$$\begin{aligned}\langle 0|[\hat{\Phi}_{IN}(x), \hat{\Phi}_{IN}(y)]|0\rangle &= \langle 0|[\hat{\Phi}^+_{IN}(x)+\hat{\Phi}^-_{IN}(x), \hat{\Phi}^+_{IN}(y)+\hat{\Phi}^-_{IN}(y)]|0\rangle \\ &= \langle 0|[\hat{\Phi}^+_{IN}(x), \hat{\Phi}^-_{IN}(y)]|0\rangle + \langle 0|[\hat{\Phi}^-_{IN}(x), \hat{\Phi}^+_{IN}(y)]|0\rangle \\ &= i\Delta^+(x-y) - i\Delta^+(y-x) \\ &= i\Delta(x-y) \qquad (3.38)\end{aligned}$$

ここで、$\langle 0|[\hat{\Phi}^+_{IN}(x), \hat{\Phi}^+_{IN}(y)]|0\rangle = \langle 0|[\hat{\Phi}^-_{IN}(x), \hat{\Phi}^-_{IN}(y)]|0\rangle = 0$ および、問題 3-9 を使った。

15. Campbell-Baker-Haussdorf の式

$$\exp\left(\hat{A}+\hat{B}\right) = \exp(\hat{A})\exp(\hat{B})\exp\left(-\frac{1}{2}[\hat{A},\hat{B}]\right). \tag{3.39}$$

で、実数 t とし $\hat{A}=t\hat{X},\ \hat{B}=t\hat{Y}$ として代入して式変形すると、交換子 $[\hat{A},\hat{B}]$ が c 数なので、

$$\exp(t\hat{X})\exp(t\hat{Y}) = \exp\left(t(\hat{X}+\hat{Y})+\frac{t^2}{2}[\hat{X},\hat{Y}]\right). \tag{3.40}$$

を得る。ところで

$$\frac{d}{dt}\left(\exp\left(t(\hat{X}+\hat{Y})+\frac{t^2}{2}[\hat{X},\hat{Y}]\right)\right) = \left(\hat{X}+\hat{Y}+t[\hat{X},\hat{Y}]\right)\left(\exp\left(t(\hat{X}+\hat{Y})+\frac{t^2}{2}[\hat{X},\hat{Y}]\right)\right). \tag{3.41}$$

これに (3.40) を代入すると、

$$\begin{aligned}\frac{d}{dt}\left(\exp(t\hat{X})\exp(t\hat{Y})\right) &= \hat{X}\exp(t\hat{X})\exp(t\hat{Y}) + \exp(t\hat{X})\hat{Y}\exp(t\hat{Y}) \\ &= \left(\hat{X}+\hat{Y}+t[\hat{X},\hat{Y}]\right)\left(\exp(t\hat{X})\exp(t\hat{Y})\right).\end{aligned} \tag{3.42}$$

右から $\exp(-t\hat{Y})$ をかけて

$$\hat{X}\exp(t\hat{X}) + \exp(t\hat{X})\hat{Y} = \left(\hat{X}+\hat{Y}+t[\hat{X},\hat{Y}]\right)\exp(t\hat{X}) \tag{3.43}$$

両辺整理すると

$$\exp(t\hat{X})\hat{Y} - \hat{Y}\exp(t\hat{X}) = [\exp(t\hat{X}),\hat{Y}] = t[\hat{X},\hat{Y}]\exp(t\hat{X}) \tag{3.44}$$

$\hat{A}'=\hat{Y},\hat{B}'=t\hat{X}$ とすると、

$$[\hat{A}',\exp(\hat{B}')] = [\hat{A}',\hat{B}']\exp(\hat{B}') \tag{3.45}$$

(ここで、交換子 $[\hat{A}',\hat{B}']$ が c 数を使っている)