第1章　Conformal theories in d dimensions

1.1　Conformal group in d dimensions

1. ミスプリ

   5ページ，下から5行目

   - 誤:$\omega^{\mu}{}_{\nu}\epsilon^{\nu}\partial_{\mu}$ → 正:$\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}$
   - 誤:$b^{\mu}(x^2\partial_{\mu} - 2x^{\mu}x\cdot\partial)$ → 正:$b^{\mu}(x^2\partial_{\mu} - 2x_{\mu}x\cdot\partial)$
     なお，$x\cdot\partial \equiv x^{\alpha}\partial_{\alpha}$

2. リー代数を満たすこと：

   (a) 和:

   $$\hat{A} + \hat{B} \tag{1.1}$$

   について演算が閉じていること，

   (b) スカラー倍: $a\hat{A}$ (a は実数) について演算が閉じていること，

   (c) 交換子:

   $$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A},$$
   $$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}] \tag{1.2}$$

   について演算が閉じていること，

   (d) Jacobi の恒等式

   $$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0 \tag{1.3}$$

   を満たすことを確認すれば良い.

交換子について Jacobi の恒等式は自明.

(a) $a^\mu \partial_\mu$

i. 和：

$$a^\mu \partial_\mu + a'^\alpha \partial_\alpha = (a^\mu + a'^\mu)\partial_\mu \tag{1.4}$$

和について閉じている.

ii. スカラー倍:自明

iii. 交換子：

$$[a^\mu \partial_\mu, a'^\alpha \partial_\alpha] = a^\mu a'^\alpha [\partial_\mu, \partial_\alpha] = 0 \tag{1.5}$$

交換子について閉じている.

以上から，$a^\mu \partial_\mu$ はリー代数を満たしている.

(b) $\omega^\mu{}_\nu x^\nu \partial_\mu$

i. 和：

$$\omega^\mu{}_\nu x^\nu \partial_\mu + \omega'^\alpha{}_\beta x^\beta \partial_\alpha = (\omega^\mu{}_\nu + \omega'^\mu{}_\nu) x^\nu \partial_\mu, \tag{1.6}$$

ここで，係数は

$$\omega_{\mu\nu} + \omega'_{\mu\nu} = -(\omega_{\nu\mu} + \omega'_{\nu\mu}), \qquad (\omega_{\mu\nu} \equiv g_{\mu\beta}\omega^\beta_\nu) \tag{1.7}$$

で反対称なので，和について閉じている.

ii. スカラー倍:

iii. 交換子：

$$\begin{aligned}
&[\omega^\mu{}_\nu x^\nu \partial_\mu, \omega'^\alpha{}_\beta x^\beta \partial_\alpha] \\
&= \omega^\mu{}_\nu \omega'^\alpha{}_\beta [x^\nu \partial_\mu, x^\beta \partial_\alpha] \\
&= \omega^\mu{}_\nu \omega'^\alpha{}_\beta (x^\nu [\partial_\mu, x^\beta \partial_\alpha] + [x^\nu, x^\beta \partial_\alpha]\partial_\mu) \\
&= \omega^\mu{}_\nu \omega'^\alpha{}_\beta (x^\nu (x^\beta [\partial_\mu, \partial_\alpha] + [\partial_\mu, x^\beta]\partial_\alpha) + (x^\beta [x^\nu, \partial_\alpha] + [x^\nu, x^\beta]\partial_\alpha)\partial_\mu) \\
&= \omega^\mu{}_\nu \omega'^\alpha{}_\beta (x^\nu ([\partial_\mu, x^\beta]\partial_\alpha) + (x^\beta [x^\nu, \partial_\alpha])\partial_\mu) \\
&= \omega^\mu{}_\nu \omega'^\alpha{}_\beta (x^\nu (\delta^\beta_\mu \partial_\alpha) + x^\beta (-\delta^\nu_\alpha)\partial_\mu) \\
&= \omega^\mu{}_\nu \omega'^\alpha{}_\mu x^\nu \partial_\alpha - \omega^\mu{}_\nu \omega'^\nu{}_\beta x^\beta \partial_\mu \\
&= (\omega^\mu{}_\gamma \omega'^\delta{}_\mu - \omega^\delta{}_\nu \omega'^\nu{}_\gamma) x^\gamma \partial_\delta
\end{aligned} \tag{1.8}$$

ところで，係数部分の添字を整理したもの $(\omega^{\mu}{}_{\gamma}\omega'_{\delta_\mu} - \omega_{\delta_\mu}\omega'^{\mu}{}_{\gamma})$ の，添字 γ と δ を入れ替えると

$$\begin{aligned}
&\omega^{\mu}{}_{\delta}\omega'_{\gamma\mu} - \omega_{\gamma_\mu}\omega'^{\mu}{}_{\delta} \\
&= \omega^{\mu}{}_{\delta}(-\omega'_{\mu\gamma}) - (-\omega_{\mu\gamma})\omega'^{\mu}{}_{\delta} \\
&= \omega_{\mu\delta}(-\omega'^{\mu}{}_{\gamma}) - (-\omega^{\mu}{}_{\gamma})\omega'_{\mu\delta} \\
&= (-\omega_{\delta\mu})(-\omega'^{\mu}{}_{\gamma}) - (-\omega^{\mu}{}_{\gamma})(-\omega'_{\delta\mu}) \\
&= -(\omega^{\mu}{}_{\gamma}\omega'_{\delta_\mu} - \omega_{\delta_\mu}\omega'^{\mu}{}_{\gamma})
\end{aligned} \tag{1.9}$$

で反対称なので，交換子について閉じている．

以上から，$\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}$ はリー代数を満たしている．

(c) $\lambda x \cdot \partial$

i. 和:

$$\lambda x \cdot \partial + \lambda' x \cdot \partial = (\lambda + \lambda')x \cdot \partial \tag{1.10}$$

和について閉じている．

ii. スカラー倍:自明

iii. 交換子:

$$\begin{aligned}
&[\lambda x \cdot \partial, \lambda' x \cdot \partial] \\
&= \lambda\lambda'[x^{\mu}\partial_{\mu}, x^{\alpha}\partial_{\alpha}] \\
&= \lambda\lambda'(x^{\mu}[\partial_{\mu}, x^{\alpha}\partial_{\alpha}] + [x^{\mu}, x^{\alpha}\partial_{\alpha}]\partial_{\mu}) \\
&= \lambda\lambda'(x^{\mu}(x^{\alpha}[\partial_{\mu}, \partial_{\alpha}] + [\partial_{\mu}, x^{\alpha}]\partial_{\alpha}) + (x^{\alpha}[x^{\mu}, \partial_{\alpha}] + [x^{\mu}, x^{\alpha}]\partial_{\alpha})\partial_{\mu}) \\
&= \lambda\lambda'(x^{\mu}(\delta^{\alpha}_{\mu}\partial_{\alpha}) + (x^{\alpha}(-\delta^{\mu}_{\alpha})\partial_{\mu}) = 0
\end{aligned} \tag{1.11}$$

以上から，$\lambda x \cdot \partial$ はリー代数を満たしている．

(d) $b^{\mu}(x^2\partial_{\mu} - 2x_{\mu}x \cdot \partial)$

i. 和:

ii. スカラー倍:

iii. 交換子:

$$\begin{aligned}
&[b^{\mu}(x^2\partial_{\mu} - 2x_{\mu}x \cdot \partial), b'^{\alpha}(x^2\partial_{\alpha} - 2x_{\alpha}x \cdot \partial)] \\
&= b^{\mu}b'^{\alpha}[(x^2\partial_{\mu} - 2x_{\mu}x^{\nu}\partial_{\nu}), (x^2\partial_{\alpha} - 2x_{\alpha}x^{\beta}\partial_{\beta})] = 0
\end{aligned} \tag{1.12}$$

1.2 Conformal algebra in 2 dimensions

1.3 Constraints of conformal invariance in d dimensions

ミスプリ：(1.13) 式の 1 行目
誤: $\langle\phi_1(x_1)\cdots\phi_\nu(x_n)\rangle$
正: $\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle$

第2章　Conformal theories in 2 dimensions

2.1　Correlation functions of primary fields

1. 13 ページ，4 行目 ミスプリ
2. (2.3) 式の 2 行目 ミスプリ

   誤: $h_1\partial\bar{\epsilon}(\bar{z}_1),\quad h_2\partial\bar{\epsilon}(\bar{z}_2)$

   正: $\bar{h}_1\partial\bar{\epsilon}(\bar{z}_1),\quad \bar{h}_2\partial\bar{\epsilon}(\bar{z}_2)$
3. 式 (2.3) から大局的共形変換で式（2．4）を導く

$$\left(\left(\epsilon(z_1)\partial_{z_1}+h_1\partial\epsilon(z_1)\right)+\left(\epsilon(z_2)\partial_{z_2}+h_2\partial\epsilon(z_2)\right)\right)G^{(2)}(z_i)=0 \quad (2.1)$$

   (a) $\epsilon(z_i)=1$

$$\left(\partial_{z_1}+\partial_{z_2}\right)G^{(2)}(z_i)=0 \quad (2.2)$$

   より，$G^{(2)}(z_i)=G^{(2)}(z_1-z_2)$

   (b) $\epsilon(z_i)=z_i$

$$\left(z_1\partial_{z_1}+h_1+z_2\partial_{z_2}+h_2\right)G^{(2)}(z_i)=0 \quad (2.3)$$

   より，$G^{(2)}(z_i)=C_{12}z_{12}^{-h_1-h_2}$

   (c) $\epsilon(z_i)=(z_i)^2$

$$\begin{aligned}
&\left(z_1^2\partial_{z_1}+2h_1z_1+z_2^2\partial_{z_2}+2h_2z_2\right)G^{(2)}(z_i)\\
&=C_{12}\left(z_1^2\partial_{z_1}+2h_1z_1+z_2^2\partial_{z_2}+2h_2z_2\right)z_{12}^{-h_1-h_2}\\
&=C_{12}\left((z_1^2-z_2^2)(-h_1-h_2)z_{12}^{-h_1-h_2-1}+(2h_1z_1+2h_2z_2)z_{12}^{-h_1-h_2}\right)\\
&=C_{12}\left((z_1+z_2)(-h_1-h_2)z_{12}^{-h_1-h_2}+(2h_1z_1+2h_2z_2)z_{12}^{-h_1-h_2}\right)\\
&=C_{12}\left(h_1z_1-h_2z_1-h_1z_2+h_2z_2\right)z_{12}^{-h_1-h_2}\\
&=C_{12}(h_1-h_2)z_{12}^{-h_1-h_2+1}=0 \quad (2.4)
\end{aligned}$$

より，

i. $h_1 \neq h_2$
   $C_{12} = 0$

ii. $h_1 = h_2 = h$
   $G^{(2)}(z_i) = C_{12} z_{12}^{-2h}$

4. 大局的共形変換で式（2．5）を導く

$$\left(\left(\epsilon(z_1)\partial_{z_1} + h_1 \partial\epsilon(z_1)\right) + \left(\epsilon(z_2)\partial_{z_2} + h_2 \partial\epsilon(z_2)\right) + \left(\epsilon(z_3)\partial_{z_3} + h_3 \partial\epsilon(z_3)\right)\right) G^{(3)}(z_i) = 0 \tag{2.5}$$

(a) $\epsilon(z_i) = 1$

$$\left(\partial_{z_1} + \partial_{z_2} + \partial_{z_3}\right) G^{(3)}(z_i) = 0 \tag{2.6}$$

より，$G^{(3)}(z_i) = G^{(3)}(z_{12}, z_{23}, z_{31})$

(b) $\epsilon(z_i) = z_i$

$$\left(z_1\partial_{z_1} + h_1 + z_2\partial_{z_2} + h_2 + z_3\partial_{z_3} + h_3\right) G^{(3)}(z_i) = 0 \tag{2.7}$$

に，$G^{(3)}(z_i) = C_{abc} z_{12}^a z_{23}^b z_{31}^c$ を代入すると

$$\begin{aligned}
&C_{abc}\left(z_1\partial_{z_1} + h_1 + z_2\partial_{z_2} + h_2 + z_3\partial_{z_3} + h_3\right) z_{12}^a z_{23}^b z_{31}^c \\
&= C_{abc}\left(z_1(a z_{12}^{-1} - c z_{31}^{-1}) + h_1 + z_2(-a z_{12}^{-1} + b z_{23}^{-1}) + h_2 + z_3(-b z_{23}^{-1} + c z_{31}^{-1}) + h_3\right) z_{12}^a z_{23}^b z \\
&= C_{abc}\left(a + b + c + h_1 + h_2 + h_3\right) z_{12}^a z_{23}^b z_{31}^c = 0
\end{aligned} \tag{2.8}$$

これから，$a + b + c + h_1 + h_2 + h_3 = 0$ でなければならない．

(c) $\epsilon(z_i) = (z_i)^2$

$$\begin{aligned}
&C_{abc}\left(z_1^2\partial_{z_1} + 2h_1 z_1 + z_2^2\partial_{z_2} + 2h_2 z_2 + z_3^2\partial_{z_3} + 2h_3 z_3\right) z_{12}^a z_{23}^b z_{31}^c \\
&= C_{abc}\left(z_1^2(a z_{12}^{-1} - c z_{31}^{-1}) + 2h_1 z_1 + z_2^2(-a z_{12}^{-1} + b z_{23}^{-1}) + 2h_2 z_2 + z_3^2(-b z_{23}^{-1} + c z_{31}^{-1}) + 2h_3 \right. \\
&= C_{abc}\left((z_1^2 - z_2^2) a z_{12}^{-1} + (z_2^2 - z_3^2) b z_{23}^{-1} + (z_3^2 - z_1^2) c z_{31}^{-1} + (2h_1 z_1 + 2h_2 z_2 + 2h_3 z_3\right) z_{12}^a z_{23}^b z \\
&= C_{abc}\left((z_1 + z_2)a + (z_2 + z_3)b + (z_3 + z_1)c + 2h_1 z_1 + 2h_2 z_2 + 2h_3 z_3\right) z_{12}^a z_{23}^b z_{31}^c \\
&= C_{abc}\left(z_1(a + c + 2h_1) + z_2(a + b + 2h_2) + z_3(c + b + 2h_3)\right) z_{12}^a z_{23}^b z_{31}^c = 0
\end{aligned} \tag{2.9}$$

上記の式が成立する為には

$$a + c + 2h_1 = 0 \tag{2.10a}$$
$$a + b + 2h_2 = 0 \tag{2.10b}$$
$$c + b + 2h_3 = 0 \tag{2.10c}$$

でなければならない．これと $a + b + c = -h_1 - h_2 - h_3$ を組み合わせると

$$a = -h_1 - h_2 + h_3 \tag{2.11a}$$
$$b = -h_2 - h_3 + h_1 \tag{2.11b}$$
$$c = -h_3 - h_1 + h_2 \tag{2.11c}$$

5. 15 ページ 4 行目

“In (2.6) the cross-ratio x is defined as $x = z_{12}z_{34}/z_{13}z_{24}$. We note that this cross-ratio is annihilated by the differential operator $\sum_{i=1}^{4} \epsilon(z_i)\partial_{z_i}$...” の意味

まず．

$$\partial_{z_1} x \equiv \partial_{z_1}\left(\frac{z_{12}z_{34}}{z_{13}z_{24}}\right) = \frac{z_{34}}{z_{13}z_{24}} - \frac{z_{12}z_{34}}{(z_{13})^2 z_{24}} = \frac{x}{z_{12}} - \frac{x}{z_{13}} \tag{2.12}$$

同様にして，

$$\sum_{i=1}^{4} \epsilon(z_i)\partial_{z_i} x$$
$$= \epsilon(z_1)\left(\frac{x}{z_{12}} - \frac{x}{z_{13}}\right) + \epsilon(z_2)\left(-\frac{x}{z_{12}} - \frac{x}{z_{24}}\right) + \epsilon(z_3)\left(\frac{x}{z_{34}} + \frac{x}{z_{13}}\right) + \epsilon(z_4)\left(-\frac{x}{z_{34}} + \frac{x}{z_{24}}\right) \tag{2.13}$$

が成り立つ．

(a) $\epsilon(z_i) = 1$

$$\sum_{i=1}^{4} \partial_{z_i} x$$
$$= \left(\frac{x}{z_{12}} - \frac{x}{z_{13}}\right) + \left(-\frac{x}{z_{12}} - \frac{x}{z_{24}}\right) + \left(\frac{x}{z_{34}} + \frac{x}{z_{13}}\right) + \left(-\frac{x}{z_{34}} + \frac{x}{z_{24}}\right) = 0 \tag{2.14}$$

(b) $\epsilon(z_i) = z_i$

$$\begin{aligned}
&\sum_{i=1}^{4} z_i \partial_{z_i} x \\
&= z_1 \left(\frac{x}{z_{12}} - \frac{x}{z_{13}} \right) + z_2 \left(-\frac{x}{z_{12}} - \frac{x}{z_{24}} \right) + z_3 \left(\frac{x}{z_{34}} + \frac{x}{z_{13}} \right) + z_4 \left(-\frac{x}{z_{34}} + \frac{x}{z_{24}} \right) \\
&= (z_1 - z_2)\frac{x}{z_{12}} + (-z_1 + z_3)\frac{x}{z_{13}} + (-z_2 + z_4)\frac{x}{z_{24}} + (z_3 - z_4)\frac{x}{z_{34}} \\
&= x - x - x + x = 0 \qquad (2.15)
\end{aligned}$$

(c) $\epsilon(z_i) = z_i^2$

$$\begin{aligned}
&\sum_{i=1}^{4} z_i^2 \partial_{z_i} x \\
&= z_1^2 \left(\frac{x}{z_{12}} - \frac{x}{z_{13}} \right) + z_2^2 \left(-\frac{x}{z_{12}} - \frac{x}{z_{24}} \right) + z_3^2 \left(\frac{x}{z_{34}} + \frac{x}{z_{13}} \right) + z_4^2 \left(-\frac{x}{z_{34}} + \frac{x}{z_{24}} \right) \\
&= (z_1^2 - z_2^2)\frac{x}{z_{12}} + (-z_1^2 + z_3^2)\frac{x}{z_{13}} + (-z_2^2 + z_4^2)\frac{x}{z_{24}} + (z_3^2 - z_4^2)\frac{x}{z_{34}} \\
&= x((z_1 + z_2) - (z_1 + z_3) - (z_2 + z_4) + (z_3 + z_4)) = 0 \quad (2.16)
\end{aligned}$$

以上より cross-ratio x が, 大局的な共形変換 $\epsilon(z_i)$ について $\sum_{i=1}^{4} \epsilon(z_i)\partial_{z_i} x = 0$ が分かる.

2.2 Radial quantization and conserved charges

1. カレントと保存電荷 Q (１７ページ, 4行目)

   (a) ミスプリ
   誤：

   $$Q(t) = \int d^d x j_0(x) \qquad (2.17)$$

   正：

   $$Q(t) = \int d^{d-1} x j_0(x) \qquad \text{(積分は時刻 } x_0 \text{ での空間成分で実行)} \qquad (2.18)$$

   (b)