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Chapter 0

The Many-Body Problem for Everybody

0.0 What the many-body problem is about

The many-body problem has attracted attention ever since the philosophers of old speculated over the question of how many angels could dance on the head of a pin. In the angel problem, as in all many-body problems, there are two essential ingredients. First of all, there have to be many bodies present—many angels, many electrons, many atoms, many molecules, many people, etc. Secondly, for there to be a problem, these bodies have to interact with each other. To see why this is so, suppose the bodies did not interact. Then each body would act independently of all the others, so that we could simply investigate the behaviour of each body separately. In other words, without interaction, instead of having one many-body problem, we would have many one-body problems. Thus, interactions are essential, and in fact the many-body problem may be defined as *the study of the effects of interaction between bodies on the behaviour of a many-body system.*

(It might be noted here, for the benefit of those interested in exact solutions, that there is an alternative formulation of the many-body problem, i.e., how many bodies are required before we have a problem? G. E. Brown points out that this can be answered by a look at history. In eighteenth-century Newtonian mechanics, the three-body problem was insoluble. With the birth of general relativity around 1910 and quantum electrodynamics in 1930, the two- and one-body problems became insoluble. And within modern quantum field theory, the problem of zero bodies (vacuum) is insoluble. So, if we are out after exact solutions, no bodies at all is already too many!)

The importance of the many-body problem derives from the fact that almost any real physical system one can think of is composed of a set of interacting particles. For example, nucleons in a nucleus interact by nuclear forces, electrons in an atom or metal interact by Coulomb forces, etc. Some examples are shown schematically in Fig. 0.1. Furthermore, it turns out that in the calculation of physical properties of such systems—for example, the energy levels of the atom, or magnetic susceptibility of the metal—interactions between particles play a very important role.

It should be clear from the variety of systems in Fig. 0.1 that the many-body problem is *not* a branch of solid state, or nuclear, or atomic physics, etc. It deals rather with *general* methods applicable to *all* many-body systems.

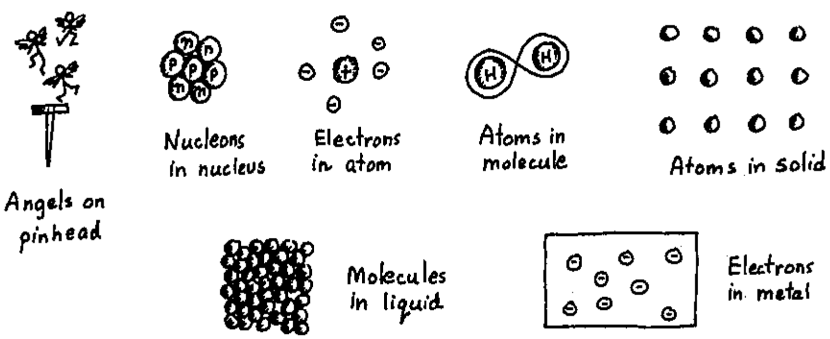

Fig. 0.1 *Some Many-body Systems*

The many-body problem is an extraordinarily difficult one because of the incredibly intricate motions of the particles in an interacting system. In Fig. 0.2 we contrast the simple behaviour of non-interacting particles with the complicated behaviour of interacting ones. Because of the complexity of the many-body problem, not much progress was made with it for a long time. In fact one of the preferred methods for solving the problem was simply to ignore it, i.e., pretend there were no interactions present. (Surprisingly enough, in some cases this 'method' produced good results anyway, and one of the great mysteries was how this could be possible!)

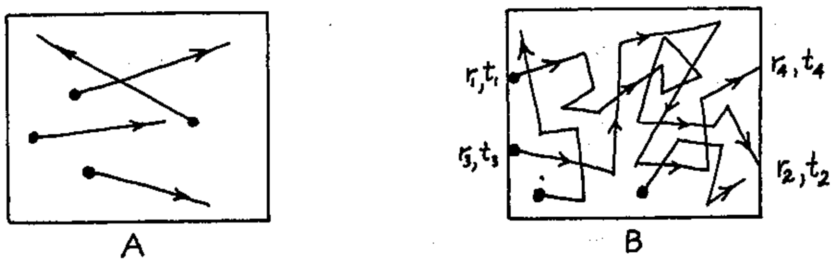

Fig. 0.2 A. *Non-interacting Particles*
B. *Interacting Particles*

Another of the early approaches to the problem, and one which is still used extensively today is the *canonical transformation* technique, described in appendix $\mathscr{A}$. This involves transforming the basic equations of the many-body system to a new set of coordinates in which the interaction term becomes small. Although considerable success has been achieved with this technique, it is not as systematic as one would like, and this sometimes makes it difficult to apply. It was this lack of a systematic method which kept the many-body field in its cradle well up into the 1950s.

The situation changed radically in 1956–7. In a series of pioneering papers, it was shown that the methods of *quantum field theory*, already famous for its success in elementary particle physics, provided a powerful, unified way of attacking the many-body problem. The new key opened many doors, and in rapid succession the idea was applied to nuclei, electrons in metals, ferromagnets, atoms, superconductors, plasmas, molecules—virtually everything in sight.

From that time on, much of the most exciting and fundamental research into the nature of matter has been based on the quantum field theory method. One of the things emerging from this research is a new simple picture of matter in which systems of interacting real particles are described in terms of approximately non-interacting fictitious bodies called 'quasi particles' and 'collective excitations'. Another thing is new results for calculated physical quantities which are in excellent agreement with experiment—for example, energy levels of light atoms, binding energy of nuclear matter, Fermi energy and effective electron mass in a variety of metals.

In this introductory chapter, we will give a physical picture of quasi particles and collective excitations. Then in the next chapter we show qualitatively how to describe quasi particles and calculate their properties by means of the quantum field theoretical technique known as the method of *Feynman diagrams*.

0.1. Simple example of non-interacting fictitious bodies

As mentioned at the beginning, one of nature's little surprises is that many-body systems often behave as if the bodies of which they are composed hardly interact at all! The reason for this is that the 'bodies' involved are not real but *fictitious*. That is, the system composed of *strongly* interacting *real* bodies acts *as if* it were composed of *weakly* interacting (or non-interacting) *fictitious* bodies. We consider now a very simple example of how this can occur.

Suppose we have two masses, m_1 and m_2 held together by a strong spring as shown in Fig. 0.3. That is, our system here consists of two strongly coupled real bodies. If this contraption is tossed up in a gravitational field, the motion of each body considered separately is very complicated because of the strong interaction (spring force) between the bodies.

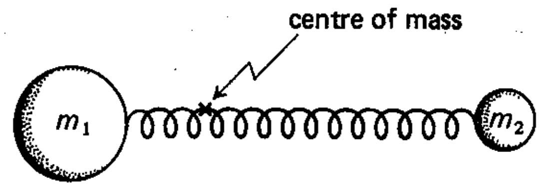

Fig. 0.3 *Two-body System*

However, we can break up the complicated motion into two independent simple motions: motion of the centre of mass and motion about the centre of mass. The centre of mass moves exactly as if it were an independent body of mass m_1+m_2, so it is one of the non-interacting fictitious bodies here. The other fictitious body is a body of mass $m_1 m_2/(m_1+m_2)$—the so-called 'reduced mass'—which moves independently relative to the centre of mass. Thus the system acts as if it were composed of two non-interacting fictitious bodies: the 'centre of mass body' and the 'reduced mass body'. (See appendix $\mathscr{A}$, eqs. ($\mathscr{A}$.11)–($\mathscr{A}$.14) for details.)

0.2 Quasi particles and quasi horses

The above two-body example is easy enough to understand, but finding the weakly interacting fictitious bodies in a set of *many* strongly interacting real bodies is a bit harder. We consider first the fictitious bodies called 'quasi particles'. These arise from the fact that when a real particle moves through the system, it pushes or pulls on its neighbours and thus becomes surrounded by a 'cloud' of agitated particles similar to the dust cloud kicked up by a galloping horse in a western. The real particle plus its cloud is the quasi particle (Fig. 0.4).

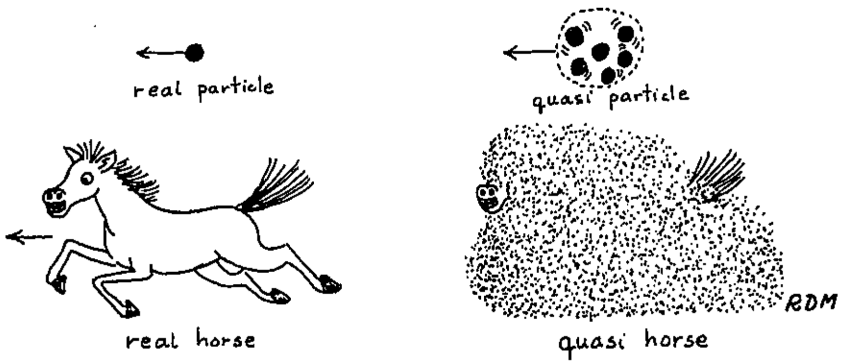

Fig. 0.4 *Quasi Particle Concept*

Just as the dust cloud hides the horse, the particle cloud 'shields' or 'screens' the real particles so that quasi particles interact only weakly with one another. The presence of the cloud also makes the properties of the quasi particle different from that of the real particle—it may have an '*effective mass*' different from the real mass, and a '*lifetime*'. These properties of quasi particles are directly observable experimentally.

It should be remarked that the quasi particle is in an excited energy level of the many-body system. Hence it is referred to as an '*elementary excitation*' of the system. (See appendix $\mathscr{A}$, §$\mathscr{A}$.2.) We now consider some examples of quasi particles.

1 *Quasi ion in a classical liquid*

Imagine that we have an electrolyte solution composed of an equal number of positive and negative ions moving about and colliding with each other as illustrated in Fig. 0.5. Let us focus our attention on a typical (+) ion in the

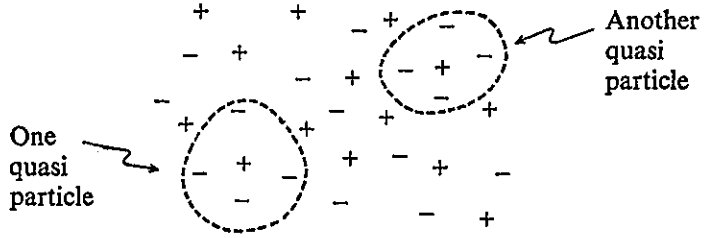

Fig. 0.5 *Quasi Particles in a Liquid of Positive and Negative Ions*

system. As this ion moves, on account of the strong Coulomb interaction, it will attract (−) ions to it. Some of these (−) ions will stick to the (+) for a while, then fall off due to collisions, then be replaced by other (−) ions, etc. Thus, on the average, because of the interaction, this typical (+) ion (and therefore every (+) ion) will be surrounded by a 'coat' or 'cloud' of (−) ions as shown in Fig. 0.5 inside the dotted lines. And of course each (−) ion will similarly have a coat of (+) ions. This coat of opposite charge will shield the ion's own charge so that its interaction with other similarly shielded ions will be much weaker than in the unshielded case. Thus the ions wearing their coats will act approximately independently of each other and constitute the quasi particles of this particular system. Many different types of systems of interacting particles may be described in this manner, and in general we have

$$\text{real particle} + \begin{matrix}\text{'coat' or 'cloud'}\\ \text{of other particles}\end{matrix} = \text{quasi particle.} \qquad (0.1)$$

Sometimes this same equation is stated in a more powerful terminology coming from quantum field theory:

$$\text{'bare' particle} + \begin{matrix}\text{'clothing'}\\ \text{or 'cloud'}\end{matrix} = \begin{matrix}\text{'dressed' or 'clothed'}\\ \text{or 'physical' or}\\ \text{'renormalized' particle.}\end{matrix} \qquad (0.2)$$

For example, in quantum electrodynamics a 'bare' electron interacting with a field of photons acquires a cloud of virtual photons around it, converting it into the 'dressed' electron. In a similar manner, the interaction between real particles is called the 'bare' interaction, while the weak interaction between quasi particles is referred to as the 'effective' or 'dressed' or 'renormalized' interaction.

It should be noted that each bare particle is simultaneously the 'core' of a quasi particle and a transient 'member' of the cloud of several other quasi particles. Therefore, if we try to visualize the whole system here as composed of quasi particles, we have to be careful, since each particle will have been counted more than once. For this reason, the quasi particle concept is valid only if one talks about a few quasi particles at a time, i.e., few in comparison with the total number of particles. In order to avoid this problem and concentrate attention on just a single quasi particle at a time, it is convenient to define quasi particles in terms of an experiment in which one adds an extra particle to the system, and observes the behaviour of this extra particle as it moves through the system. This is shown in Fig. 0.6 for a (+) ion.

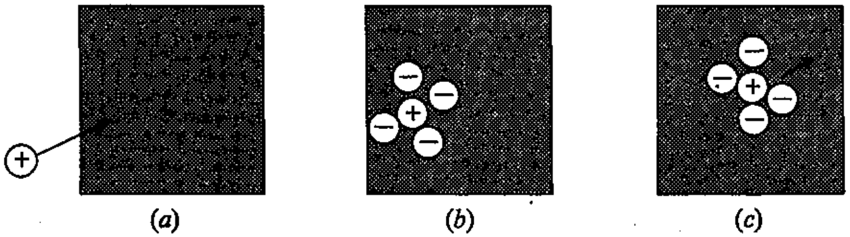

Fig. 0.6 *Moving Quasi Ion. (a) Extra (+) Ion Shot into Liquid. (b) (+) Ion Acquires Cloud of (−) Ions, Turning it into Quasi Ion. (c) Quasi Ion Moves Through System*

With this intuitive picture in mind, it is possible to guess at some of the properties of quasi particles. First, because there is in general still a small interaction left between quasi particles, a quasi particle of momentum **p** will only keep this momentum for an average time τ_p. This can be understood from Figs. 0.6 and 0.5. If the quasi ion in Fig. 0.6 (*b*) has momentum **p**, it will propagate undisturbed an average time τ_p before undergoing a collision with another quasi ion in the system (that is, a quasi ion which *belongs* to the system, like those shown in Fig. 0.5, *not* one which we shoot into the system) which scatters it out of momentum state **p**. Hence

$$\text{quasi particles have a lifetime, } \tau_p. \tag{0.3}$$

The lifetime must be reasonably long for us to say that the quasi particle approximation is a good one. It can also be seen that because of the average coat of particles on its back, the quasi particle may have an 'effective' or 'renormalized' mass which is different from that of the bare particle. (The effective mass concept is not always applicable however.) This implies that free quasi particles (i.e., not in an externally applied field) have a new energy law

$$\epsilon' = \frac{p^2}{2m^*} \quad \text{instead of} \quad \epsilon = \frac{p^2}{2m} \tag{0.4}$$

where m^* is the effective mass. The difference

$$\epsilon_{\substack{\text{quasi}\\ \text{particle}}} - \epsilon_{\substack{\text{bare}\\ \text{particle}}} = \epsilon_{\text{self}} \tag{0.5}$$

is called the '*self-energy*' of the quasi particle. This comes from the interpretation that the bare particle interacts with the many-body system, creating the cloud, and the cloud in turn reacts back on the particle, disturbing its motion. Thus the particle is, in a sense, interacting with itself via the many-body system, and changing its own energy.

2 *Quantum system: quasi electron in electron gas*

The 'electron gas' is a simple model often used to describe many-body effects in metals. It consists of a box containing a large number of electrons interacting by means of the Coulomb force. In addition, there is a uniform, fixed, positive charge 'background' put into the box in order to keep the whole system electrically neutral. In the ground state, the electrons are spread out uniformly in the box, as shown schematically in Fig. 0.7.

Fig. 0.7 *'Electron Gas': Interacting Electrons Spread Out Uniformly in Box, plus Uniform, Fixed, Positive Charge Background*

Suppose now that we have a single, well-localized electron which we shoot into the electron gas (Fig. 0.8). Because of the repulsive Coulomb interaction between electrons, this extra electron repels other electrons away from it, so

Fig. 0.8 *Extra Electron Shot into Electron Gas*

we get an 'empty space' near the extra electron, and repelled electrons further away (Fig. 0.9). The empty space has positive charge, since the positive charge background is exposed in this region. This empty region may be viewed in a more detailed or 'microscopic' way as composed of 'holes' in the electron gas. That is, the extra electron has 'lifted out' electrons from the uniform charge distribution in its vicinity, thus creating 'holes' in this charge distribution, and has 'put down' these lifted-out electrons further away. This is shown in Fig. 0.10. Because of the exposed positive background, these holes have positive charge.

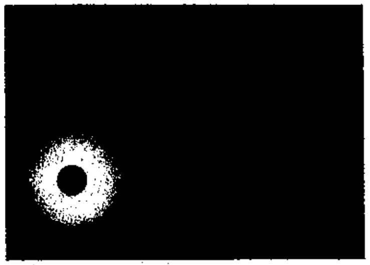

Fig. 0.9 *Extra Electron Pushes Other Electrons Away, Creating 'Empty' Region in its Immediate Vicinity*

Fig. 0.10 *'Microscopic' View of Fig. 0.9 Showing Electrons Lifted out from Vicinity of the Extra Electron, thus Creating 'Holes'*

The above definition of hole in the sense of 'empty place' is the one commonly used in solid state physics. However, later on we shall re-define things so that the hole becomes an 'anti-particle' analogous to those of elementary particle physics (see §4.2).

The holes and lifted out electrons are constantly being destroyed by interaction with the extra electron and with the other electrons in the system, and new holes and lifted out electrons take their place. The sum of these microscopic processes, which go on all the time, is Fig. 0.9. Thus Fig. 0.9 may be visualized as an extra electron surrounded by a 'cloud' of constantly changing holes and lifted out electrons. This combination is called the *quasi electron.*

The quasi electron moves or 'propagates' through the system as shown in Fig. 0.11.

We now notice that the positive hole cloud immediately around the extra electron partially shields the electron's own negative charge. Hence, if we have two quasi electrons as shown in Fig. 0.12, and these are far enough

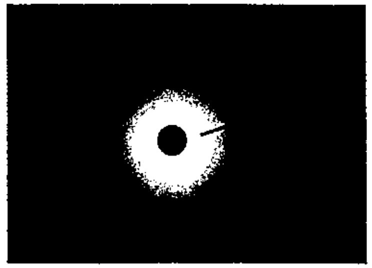

Fig. 0.11 *Quasi Electron Propagates Through System*

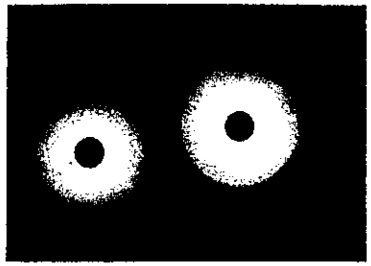

Fig. 0.12 *Two Quasi Electrons Interact only Weakly Because of Shielding*

apart so that their clouds do not overlap very much, then we see that because of the shielding the two quasi electrons will interact only weakly. That is, quasi electrons act nearly independently of one another. This is why metals generally behave as if their electrons were independent: it is not real electrons but rather quasi electrons we are looking at.

3 *Single electron in a metal*

Actually, the simplest quantum example of the quasi particle idea occurs not in a true many-body system, but rather in a system containing one particle moving in an external potential, i.e., a conduction electron in a metal. In a perfect metal the positive ions form a regular periodic lattice (we ignore lattice vibrations for the moment) so that the electron moves in a periodic force field due to the attractive Coulomb interaction between the ions and the electron (see Fig. 0.13a). In an imperfect metal, the periodicity is spoiled by the presence of a more or less random distribution of some impurity ions in the lattice, or the presence of some displaced ions (Fig. 0.13b).

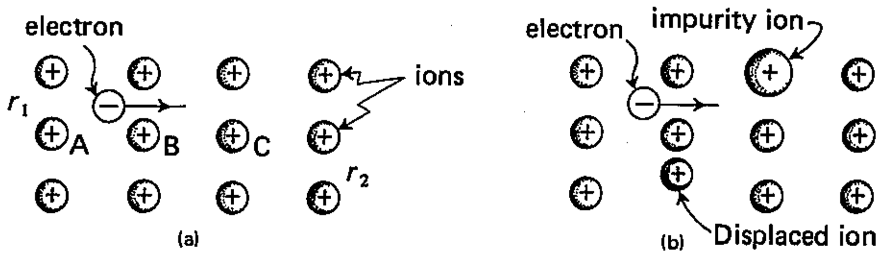

Fig. 0.13 (*a*) *Conduction Electron in Perfect Metal.* (*b*) *Imperfect Metal*

Since the lattice here is assumed fixed, there is no 'moving cloud' of lattice ions following the electron. Nevertheless, it turns out that even these stationary lattice ions are capable of 'clothing' the electron, and we find that for a perfect lattice, there is an effective mass, m^*, and an infinite lifetime. Addition of imperfections causes the lifetime to become finite.

4 *Quasi nucleon*

Despite powerful short-range forces between nucleons in a nucleus, they behave in many respects as if they were independent of each other, as is indicated by the success of the nuclear shell model. The nearly independent particles here are not the nucleons themselves, but the nucleons each surrounded by a cloud of other nucleons, i.e., the quasi nucleons.

5 *Bogoliubov quasi particles* ('*bogolons*')

These are the elementary excitations in a superconductor. We include them here since they are called quasi particles, but actually their structure is quite different from the 'particle plus cloud' picture described above. They consist of a linear combination of an electron in state $(+k, \uparrow)$ and a 'hole' in $(-k, \downarrow)$.

0.3 Collective excitations

As we have seen, the quasi particle consists of the original real, individual particle, plus a cloud of disturbed neighbours. It behaves very much like an individual particle, except that it has an effective mass and a lifetime. But there also exist other kinds of fictitious particles in many-body systems, i.e., 'collective excitations'. These do not centre around individual particles, but instead involve collective, wavelike motion of *all* the particles in the system simultaneously. Here are some examples:

1 *Plasmons*

If a thin metal foil is bombarded with high energy electrons, it is possible to set up sinusoidal oscillations in the density of the electron gas in the foil. This is known as a 'plasma wave', and it has a frequency ω_p and a wavelength λ_p (see Fig. 0.14a). The plasma wave may be visualized as built up of 'holes'

in the low-density regions and extra electrons in the high-density regions as shown in Fig. 0.14(*b*). Just as light waves are quantized into units having energy $E=\hbar\omega$ called photons, plasma waves are quantized into units with energy $E_p=\hbar\omega_p$ called *plasmons.*

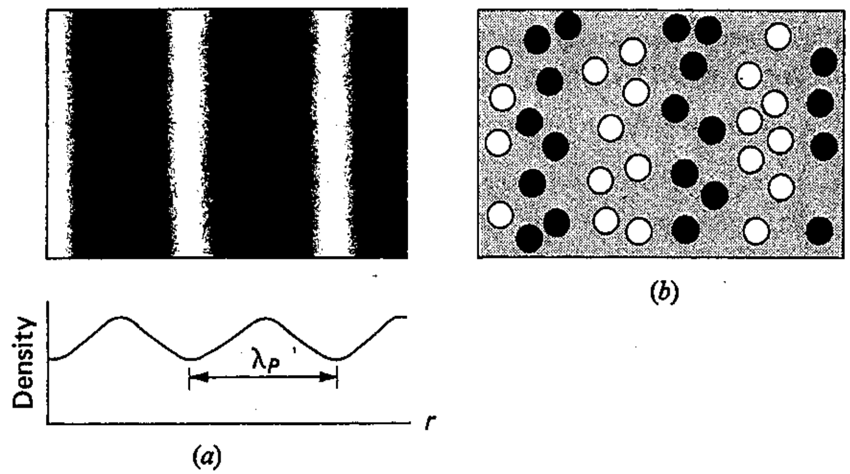

Fig. 0.14 (*a*) *Plasma Wave in Electron Gas.* (*b*) *Particle-hole Picture of Plasma Wave*

2 *Phonons*

Sound waves are sinusoidal oscillations in the crystal lattice of a solid. They are quantized into collective excitations called 'phonons'. (See appendix $\mathscr{A}$.)

3 *Magnons*

In ferromagnets there are regular fluctuations in the density of spin angular momentum known as 'spin waves'. The collective excitation here is the spin wave quantum known as the 'magnon'.

4 *Nuclear quanta*

In nuclei, one finds various vibrational and rotational motions; the associated quanta are the collective excitations in this case.

In the next chapter, we will describe in a very qualitative way how to find the properties of quasi particles and collective excitations by means of 'propagators' and 'Feynman diagrams'.

Further reading

Appendix $\mathscr{A}$
Patterson (1964).
Pines (1963), chap. 1.

Chapter 1

Feynman Diagrams, or how to Solve the Many-Body Problem by means of Pictures

1.1 Propagators—the heroes of the many-body problem

We have seen that many-body systems consisting of strongly interacting real particles can often be described as if they were composed of weakly interacting fictitious particles: quasi particles and collective excitations. The question now is, how can we calculate the properties of these fictitious particles—for example, the effective mass and lifetime of quasi particles? There are various ways of doing this (see appendix $\mathscr{A}$) but the hero roles in the treatment of the many-body problem are played by quantum field theoretical quantities known as *Green's functions* or *propagators*. These are essentially a generalization of the ordinary, familiar undergraduate Green's function. They come in all sizes and shapes—one particle, two particle, no particle, advanced, retarded, causal, zero temperature, finite temperature—an assortment to suit every situation and taste.

There are three reasons for the immense popularity propagators are enjoying these days. First of all, they yield in a direct way the most important physical properties of the system. Secondly, they have a simple physical interpretation. Thirdly, they can be calculated in a way which is highly systematic and 'automatic' and which appeals to one's physical intuition.

The idea behind the propagator method is this: the detailed description of a many-body system requires in the classical case the position of each particle as a function of time, $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, …, $\mathbf{r}_N(t)$, or in the quantum case, the time-dependent wave function of the whole system, $\Psi(\mathbf{r}_1 . \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$. A glance at Fig. 0.2B shows that this is an extremely complicated business. Fortunately, it turns out that in order to find the important physical properties of a system it is not necessary to know the detailed behaviour of each particle in the system, but rather just the *average* behaviour of one or two typical particles. The quantities which describe this average behaviour are the *one-particle propagator* and *two-particle propagator* respectively, and physical properties may be calculated directly from them.

Consider the one-particle propagator first. It is defined as follows: We put a particle into the interacting system at point r_1 at time t_1 and let it move through the system colliding with the other particles for a while (i.e., let it 'propagate'

through the system). Then the one-particle propagator is the probability (or in quantum systems, the probability *amplitude*—see §3.1) that the particle will be observed at the point r_2 at time t_2. (Note that instead of putting the particle in at a definite point, it is sometimes more convenient to put it in with definite momentum, say p_1, and observe it later with momentum p_2.) The single-particle propagator yields directly the energies and lifetimes of quasi particles. It also gives the momentum distribution, spin and particle density and can be used to calculate the ground state energy.

Similarly, the two-particle propagator is the probability amplitude for observing one particle at r_2, t_2 and another at r_4, t_4 if one was put into the system at r_1, t_1 and another at r_3, t_3 (see Fig. 0.2B). This also has a wide variety of talents, giving directly the energies and lifetimes of collective excitations, as well as the magnetic susceptibility, electrical conductivity, and a host of other non-equilibrium properties.

There is also another useful quantity, the 'no-particle propagator' or so-called 'vacuum amplitude' defined thus: We put no particle into the system at time t_1, let the particles in the system interact with each other from t_1 to t_2, then ask for the probability amplitude that no particles emerge from the system at time t_2. This may be used to calculate the ground state energy and the grand partition function, from which all equilibrium properties of the system may be determined.

1.2 Calculating propagators by Feynman diagrams: the drunken man propagator

There are two different methods available for calculating propagators. One is to solve the chain of differential equations they satisfy—this method is discussed briefly in appendix M. The other is to expand the propagator in an infinite series and evaluate the series approximately. This can be carried out in a general, systematic, and picturesque way with the aid of *Feynman diagrams*.

Just to get an idea of what these diagrams are, consider the following simple example (see Fig. 1.1). A man who has had too much to drink, leaves a party at point 1 and on the way to his home at point 2, he can stop off at one or more bars—Alice's Bar (A), Bardot Bar (B), Club Six Bar (C), …, etc. He can wind up either at his own home 2, or at any one of his friends' apartments, 3, 4, etc. We ask for the probability, $P(2,1)$, that he gets home. This probability, which is just the propagator here (with time omitted for simplicity), is the sum of the probabilities for all the different ways he can propagate from 1 to 2 interacting with the various bars.

The first way he can propagate is 'freely' from 1 to 2, i.e., without stopping at a bar. Call the probability for this free propagation $P_0(2,1)$.

The second way he can propagate is to go freely from 1 to bar A (the probability for this is $P_0(A,1)$), then stop off at bar A for a drink (call the probability for this $P(A)$), then go freely from A to 2 (probability$=P_0(2,A)$). Assume for simplicity that the three processes here are independent. Then the total probability for this second way is the product of the probabilities for each process taken separately, i.e., $P_0(A,1)\times P(A)\times P_0(2,A)$. (This is like the case in coin-tossing: since each toss is independent, the probability of first tossing a head, then a tail, equals the probability of tossing a head times the probability of tossing a tail.)

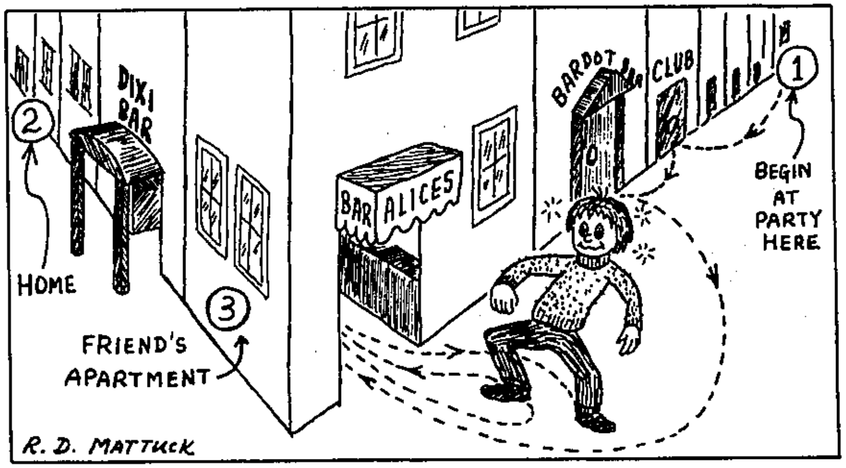

Fig. 1.1 *Propagation of Drunken Man*

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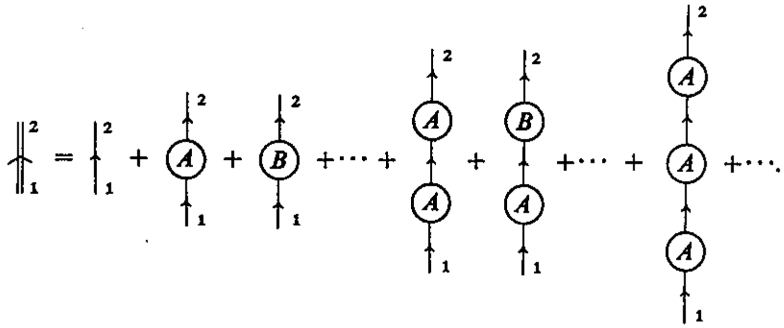
The third way he can propagate is from 1 to B to 2, with probability $P_0(B,1)P(B)P_0(2,B)$. Or he could go from 1 to C to 2, etc., or from 1 to A to B to 2, or from 1 to A, come out of A, go back into A, then go to 2, and so on. The total probability, $P(2,1)$ is then given by the sum of the probabilities for each way, i.e., the infinite series:

$$P(2,1) = P_0(2,1)+P_0(A,1)P(A)P_0(2,A)+P_0(B,1)P(B)P_0(2,B)+\cdots$$
$$+P_0(A,1)P(A)P_0(B,A)P(B)P_0(2,B)+\cdots. \tag{1.1}$$

This is an example of a 'perturbation series', since each interaction with a bar 'perturbs' the free propagation of the drunken man.

Now, such a series is a complicated thing to look at. To make it easier to read, we follow the journal 'Classic Comics' where difficult literary classics are translated into picture form. Let us make a 'picture dictionary' to associate

diagrams with the various probabilities as in Table 1.1. Using this dictionary, series (1.1) can be drawn thus:



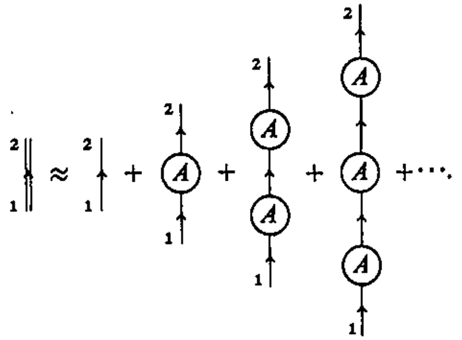
(1.2)

Since, by dictionary Table 1.1, each diagram element stands for a factor, series (1.2) is completely equivalent to (1.1). However it has the great advantage that it also reveals the physical meaning of the series, giving us a 'map' which helps us to keep track of all the sequences of interactions with bars which the drunken man can have in going from 1 to 2.

Table 1.1 *Diagram dictionary for drunken man propagator*

Word	Picture	Meaning
$P(2,1)$		probability of propagation from 1 to 2
$P_0(s,r)$		probability of free propagation from r to s
$P(X)$		probability of stopping off at bar X for a drink

The series may be evaluated approximately by selecting the most important types of terms in it and summing them to infinity. This is called *partial summation.* For example, suppose the man is in love with Alice, so that $P(A)$ is large, and all the other $P(X)$'s are small. Then Alice's bar diagrams will dominate, and the series (1.2) may be approximated by a sum over just repeated interactions with Alice's Bar:



(1.3)

Using the above dictionary, this can be translated into functions:

$$P(2,1) \approx P_0(2,1)+P_0(A,1)P(A)P_0(2,A)+ \\ +P_0(A,1)P(A)P_0(A,A)P(A)P_0(2,A)+\cdots. \quad (1.4)$$

Assume for simplicity that all $P_0(s,r)$ are equal to the same number, c, i.e., $P_0(2,1)=P_0(2,A)=P_0(A,1)=P_0(A,A)=c$. Then series (1.4) becomes

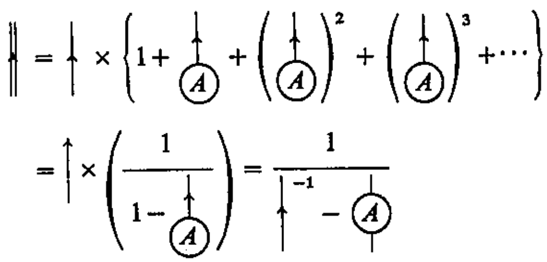
$$P(2,1) = c+c^2P(A)+c^3P^2(A)+\cdots \\ = c\{1+cP(A)+[cP(A)]^2+[cP(A)]^3+\cdots\}. \quad (1.5)$$

The series in brackets is geometric and can be summed exactly to yield $1/(1-cP(A))$, so that

$$P(2,1) = c\times\left(\frac{1}{1-cP(A)}\right)=\frac{1}{c^{-1}-P(A)} \quad (1.6)$$

which is the solution for the propagator in this case.

Note that since each diagram element stands for a factor, we could have done calculation (1.5), (1.6) completely diagrammatically:



(1.7)

The partial summation method is extremely useful in dealing with the strong interactions between particles in the many-body problem, and it is the basic method which will be used throughout this book.

1.3 Propagator for single electron moving through a metal

The example here is just like the previous one, except that instead of a propagating drunken man interacting with various bars, we have a propagating electron interacting with various ions in a metal. A metal consists of a set of positively charged ions arranged so they form a regular lattice, as in Fig. 0.13A or a lattice with some irregularities, as in Fig. 0.13B. An electron interacts with these ions by means of the Coulomb force. The single particle propagator here is the sum of the *quantum mechanical probability amplitudes* (see §3.1) for all the possible ways the electron can propagate from point r_1 in the crystal at time t_1, to point r_2 at time t_2, interacting with the various ions on the way. These are: (1) freely, without interaction; (2) freely from r_1, t_1 (='1' for short) to the ion at r_A at time t_A, interaction with this ion, then free propagation from the ion to point 2; (3) from 1 to ion B, interaction at B, then from B to 2, etc. Or we could have the routes 1–A–A–2, 1–A–B–2, etc. We can now use the dictionary in Table 1.1 to translate this into diagrams, provided the following changes are made: change 'probability' to 'probability amplitude', and change the meaning of the circle with an X to 'probability amplitude for an interaction with the ion at X'. When this is done, the series for the propagator can be translated immediately into exactly the same diagrams as in the drunken man case! That is, (1.2) is also the propagator for an electron in a metal, provided that we just use a quantum dictionary to translate the lines and circles into functions. The series can be partially summed, and from the resulting propagator we obtain immediately the energy of the electron moving in the field of the ions.

1.4 Single-particle propagator for system of many interacting particles

We will now indicate in a qualitative way how the single-particle propagator may be calculated in a system of many interacting particles. The argument is general, but we may think in terms of the electron gas as illustration. The propagator will be the sum of the probability amplitudes for all the different ways the particle can travel through the system from r_1, t_1 to r_2, t_2. First we have free propagation without interaction. Another thing which can happen is shown in the 'movie', Fig. 1.2, which depicts a 'second-order' propagation process (i.e., a process with *two* interactions). (It should be mentioned here that unlike the drunken man case, the processes involved in Fig. 1.2 are not real physical processes, but rather 'virtual' or 'quasi physical', since they do not conserve energy, and they may violate the Pauli exclusion principle. The reason for this is that, as we shall see later on, the sequence in Fig. 1.2 (or the corresponding diagram (1.9)) is simply a convenient and picturesque way of describing a certain second-order term which appears in the perturbation expansion of the propagator. Hence Fig. 1.2 and diagram (1.9) are in reality *mathematical expressions* so we have to be careful not to push their physical interpretation too far (see §4.6).)

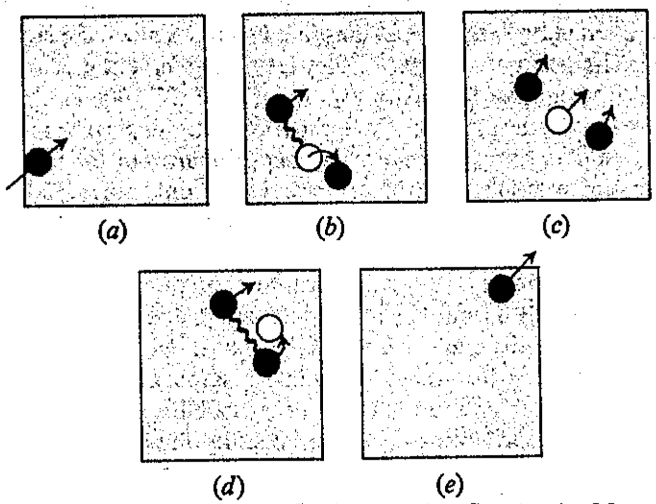
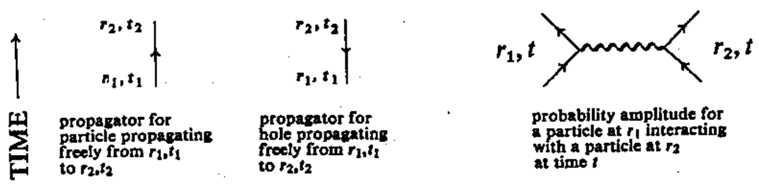


Fig. 1.2 *'Movie' of Second-order Propagation Process in Many-body System*

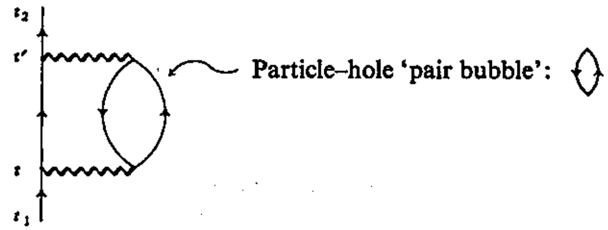
(*a*) At time t_1, extra particle enters system.
(*b*) At time t, extra particle interacts (wavy line) with a particle in the system, lifting it out of its place, thus creating a 'hole' in the system.
(*c*) The extra particle, plus the 'hole' and the 'lifted-out' particle ('particle–hole pair') travel through the system.
(*d*) At time t', the extra particle interacts with the 'lifted-out' particle, knocking it back into the hole, thus destroying the particle–hole pair.
(*e*) At time t_2, the extra particle moves out of the system.

To represent this sequence of events diagrammatically, let us imagine that time increases in the upward-going direction and we use the following diagram elements:



(1.8)

(Note that the hole is drawn as a particle moving backward in time. The reason for this is in §4.2.) Then the probability amplitude for the above sequence of events can be represented by the diagram



(1.9)

The piece of diagram:

(1.10)

is called a 'self-energy part' because it shows the particle interacting with itself via the particle–hole pair it created in the many-body medium. Diagram (1.9) may be evaluated by writing a free propagator factor for each directed line, and an amplitude factor for each wiggly line (see Chapter 4, Table 4.3), analogous to the drunken man case.

Another sequence of events which can occur involves only one interaction (i.e., a 'first-order' process). It is a quick-change act in which the incoming electron at point r interacts with another electron at point r' and changes place with it. This is analogous to billiard ball 1 striking billiard ball 2 and transferring all its momentum to 2. The first-order process and its analogy are shown in Fig. 1.3. The sequence may be drawn diagrammatically

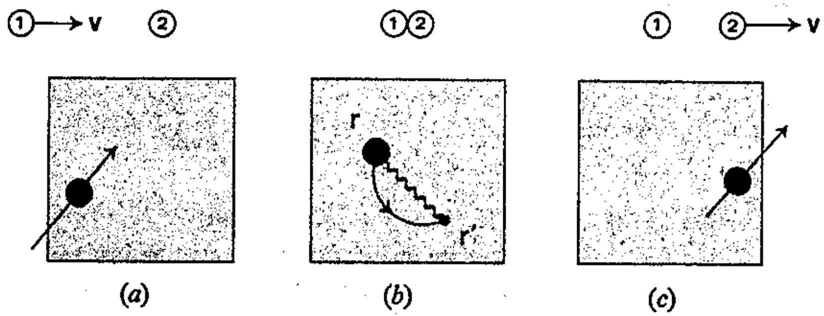
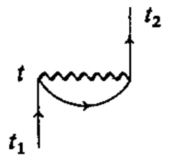


Fig. 1.3 *Movie of First-order Process (Lower Drawing) and its Analogy (Upper Drawing)*

(*a*) Extra particle enters at time t_1.
(*b*) At time t, the particle is at point r. It interacts with a particle at r' and changes place with it.
(*c*) Extra particle leaves at time t_2.

as in (1.11):

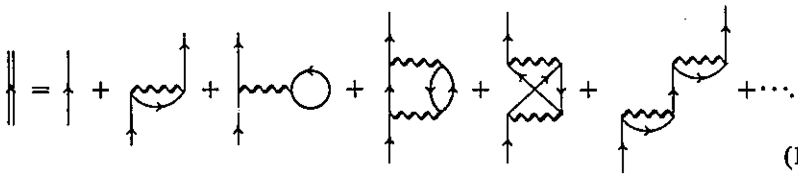


(1.11)

'Open oyster' diagram
(closed oyster is in (1.19))

The diagrams in (1.8)–(1.11) are called *Feynman diagrams* after their inventor, Richard P. Feynman who employed them in his Nobel prize-winning work on quantum electrodynamics. They are used extensively in elementary particle physics.

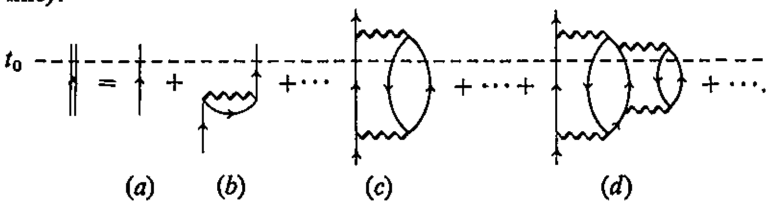
The total single particle propagator is the sum of the amplitudes for all possible ways the particle can propagate through the system. This will include the above processes, repetitions of them, plus an infinite number of others. Thus we find



(1.12)

(Note: the interpretation of the 'bubble' diagram, just after the open oyster, will be discussed in chapter 4.)

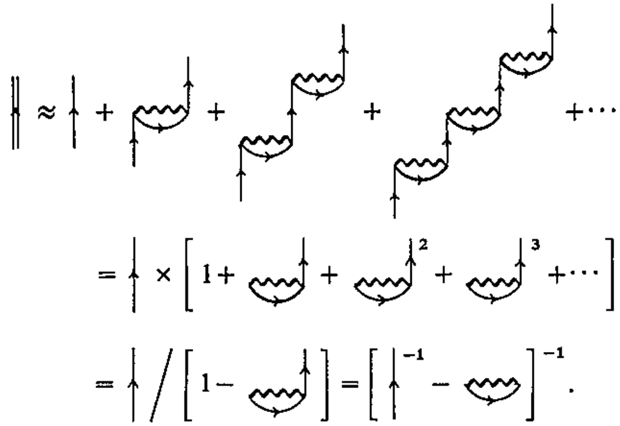
We can see the direct connection between the one-particle propagator and the quasi particle by looking at all the diagrams at a particular time t_0 (dashed line):



(1.13)

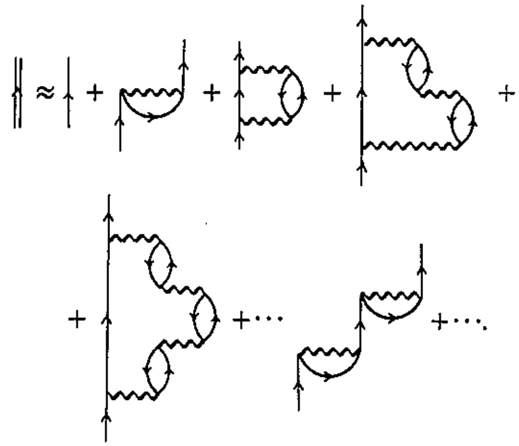
At t_0, we see that various situations may exist: there may be just the bare particle (*a*), or there may exist two particles plus one hole created by the second-order sequence (*c*), or three particles plus two holes in (*d*), etc. That is, the diagrams show all the configurations of particles and holes which may be kicked up by the bare particle as it churns through the many-body system. If we now compare with the picture of the quasi particle in Fig. 0.10, we see that *the diagrams reveal the content of the ever-changing cloud of particles and holes surrounding the bare particle and converting it into a quasi particle.*

Just as in the drunken man case, the propagator here may be calculated approximately by doing a partial sum. For example, we can sum over all diagrams containing repeated open oyster parts since they constitute a geometric series (cf. (1.7)):



(1.14)

For the electron gas, this is the 'Hartree-Fock' approximation. We can also include 'ring' diagrams in the sum, i.e., diagrams in which the self-energy parts are composed of rings of particle–hole pair bubbles (these are the most important in a high-density electron gas):



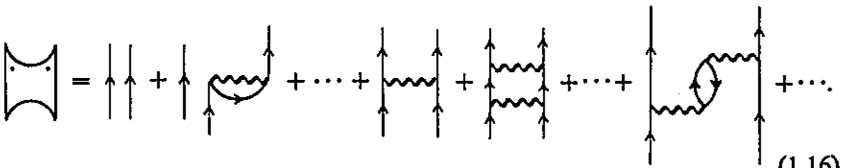
(1.15)

This sum can be carried out and yields the so-called 'random phase approximation' or 'RPA', which is extremely useful in analysing the properties of metals.

Note that the essential thing involved in the above partial sums is the structure or *topology* of the diagrams, i.e., how the various lines are connected to each other. Thus we could sum (1.14) because each diagram consisted of single lines connecting the same repeated part. This diagram topology is the key to the quantum field theoretical method in the many-body problem.

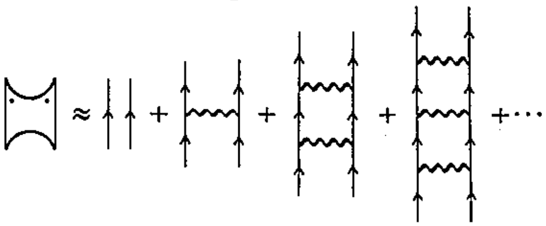
1.5 The two-particle propagator and the particle-hole propagator

The two-particle propagator is the sum over the probability amplitudes for all the ways two particles can enter the system, interact with each other and with the particles in the system, then emerge again. The diagram series for it is (note that the dots on the diagram for the two-particle propagator show the points at which directed lines emerge):



(1.16)

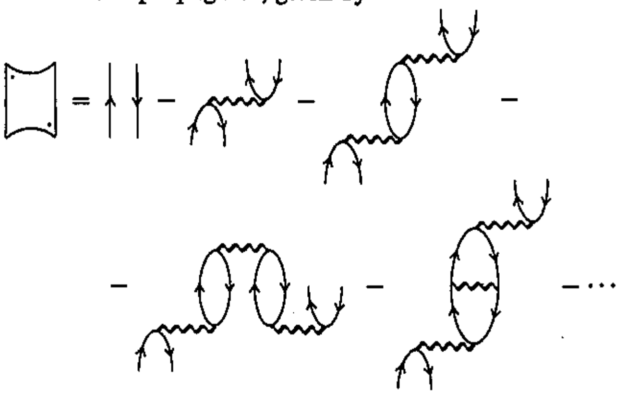
A partial sum over all 'ladder' diagrams here:



(1.17)

is called 'ladder' approximation, and is very useful in describing nuclear matter, and low-density systems.

The 'particle-hole' propagator, given by



(1.18)

may be used to find the energy and lifetime of collective excitations, e.g. plasmons.

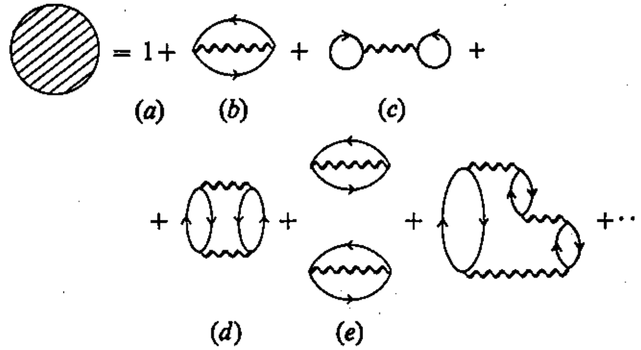
1.6 The no-particle propagator ('vacuum amplitude')

The ground state energy of a many-body system may be obtained directly from the no-particle propagator, or 'vacuum amplitude'. This is the sum of amplitudes for all the ways the system can begin at time t_1 with no extra or lifted-out particles, or holes in it (this is the undisturbed or 'Fermi vacuum' state), have its particles interact with each other, and wind up at t_2 with no extra or lifted-out particles, or holes. The simplest process is where nothing at all happens—the system just sits there. A first-order process occurs in which two particles change places with each other as shown in the following diagram

(1.19)

'Oyster' diagram

A more complicated process is shown in Fig. 1.4. The vacuum amplitude may thus be represented by the following diagram series:



(1.20)

where '1' is for the nothing-at-all process and (*d*) is the picture for Fig. 1.4. (The 'double bubble' diagram, (*c*), is discussed in chapter 5.)

The vacuum amplitude series gives us a vivid picture of the ground state of the many-body system as a sort of 'virtual witches' brew', constantly seething, with particles and holes boiling up, bubbling, and colliding, as in Fig. 1.5.

In conclusion, we see that Feynman diagrams have many appealing features, besides their utility as a calculational tool. One thing which was already pointed out in §1.2 is the fact that they show directly the physical meaning of the perturbation term they represent. Another thing is that they reveal at a glance the structure of very complicated approximations by showing which sets of diagrams have been summed over. In this way, they have introduced a new language into physics, and one often sees phrases like 'ladder approximation' or 'ring approximation' even in articles in which no diagrams appear. And finally, one cannot be immune to the Klee-like charm of the diagrams. Includ-

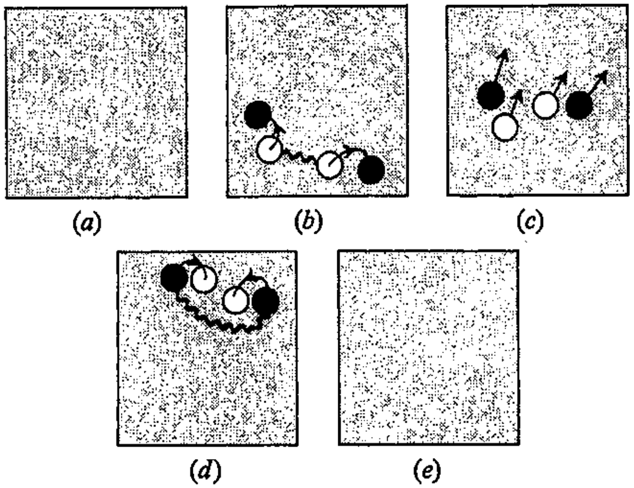


Fig. 1.4 *Virtual Movie of Second-order Vacuum Amplitude Process*
(*a*) Vacuum.
(*b*) At time t_1 interaction between two particles in system causes two particles to be lifted out, forming two holes.
(*c*) The two particle–hole pairs propagate freely through the system.
(*d*) Both pairs annihilated at time t'.
(*e*) Vacuum.

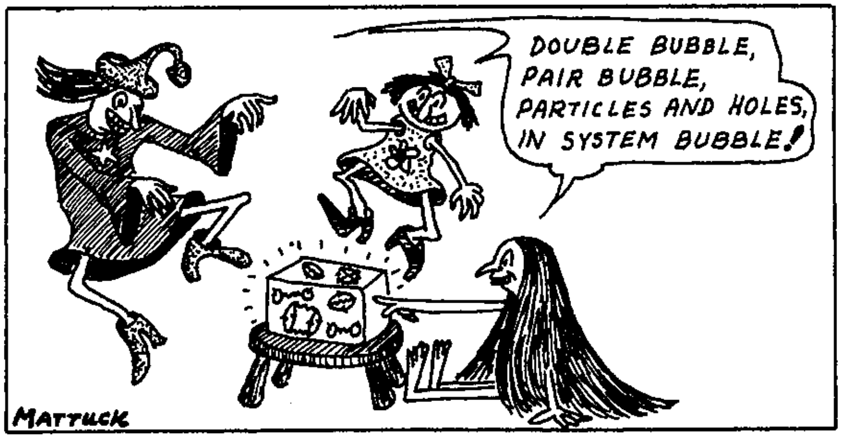


Fig. 1.5 *Modern View of a Many-body System in its Ground State*

ing in their ranks, in addition to the above, such characters as the 'necklace', the 'potato' and the 'tadpole', plus infinite numbers yet unnamed, they constitute what might indeed be called 'perturbation theory in comic-book form.'

Chapter 2

Classical Quasi Particles and the Pinball Propagator

2.1 Physical picture of quasi particle

We saw in §0.2 that the quasi particle is one type of elementary excitation in a many-body system, and that physically it consists of a particle surrounded by a cloud of other particles. The concept was illustrated by examples ranging from the quasi electron to the quasi horse. We also saw how quasi particles may be described by means of propagators, which are calculated with the aid of Feyman diagrams. Here we start with a brief review of the quasi particle idea, then go on to describe the form of the propagator for a classical quasi particle. The partial sum method of calculating the classical propagator is discussed in detail with the aid of a pinball machine example.

For concreteness, let us think in terms of the classical quasi ion in Fig. 0.6 which consists of a bare ion plus a coat of oppositely charged ions surrounding it. This picture led us to the general definition

$$\text{real particle} + \begin{matrix}\text{'coat' or 'cloud'}\\ \text{of other particles}\end{matrix} = \text{quasi particle} \tag{2.1}$$

or

$$\text{'bare' particle} + \begin{matrix}\text{'clothing'}\\ \text{or 'cloud'}\end{matrix} = \begin{matrix}\text{'dressed' or 'clothed'}\\ \text{or 'renormalized'}\\ \text{particle}\end{matrix} \tag{2.2}$$

It may be remarked that if we perform a 'Gedanken' calculation and imagine that the transformation in appendix ($\mathscr{A}$.9) were carried out, we see that the quasi particle co-ordinate $\mathbf{R}_i$ will involve the real particle co-ordinate $\mathbf{r}_i$, plus the co-ordinates $\mathbf{r}_j(j \neq i)$ of all the other particles in the system. The $\mathbf{r}_j(j \neq i)$ then evidently describe the shifting cloud, so it is therefore proper to call the cloud a part of the quasi particle.

We saw also that because of the small interactions between quasi particles,

$$\text{quasi particles have a lifetime, } \tau_p, \tag{2.3}$$

and because of their coat of other particles, quasi particles have a new energy

$$\epsilon = \frac{p^2}{2m^*} \tag{2.4}$$

law where m^* is the effective mass. Finally, we defined the self-energy, ϵ_{self} by

$$\epsilon_{\substack{\text{quasi}\\\text{particle}}} - \epsilon_{\substack{\text{bare}\\\text{particle}}} = \epsilon_{self}. \tag{2.5}$$

2.2 The classical quasi particle propagator

Quasi particles in a system may be tracked down by means of the single particle Green's function or 'propagator'. Let us see what this is in the classical case. Imagine we have a many-body system, and we consider the motion of one particle in it under the influence of a constant external force $\mathbf{F}$ applied to it as shown in Fig. 2.1. Suppose the particle begins at $\mathbf{r}_1$ at time t_1.

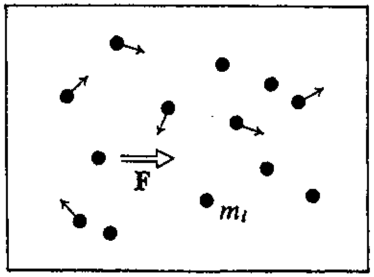


Fig. 2.1 *Many-body System*

If there are no collisions with other particles, the movement or 'propagation' of the particle to the point $\mathbf{r}_2$ at time t_2 is described by

$$\mathbf{r}_2-\mathbf{r}_1 = \frac{1}{2}\left(\frac{\mathbf{F}}{m}\right)(t_2-t_1)^2. \tag{2.6}$$

But in the interacting case, collisions take place, and the particle will follow a highly irregular path not described by (2.6). The best one can do in this situation is to talk about the *probability* of the particle going from one point to another. This leads us to define the *classical propagator*:

$P(\mathbf{r}_2, t_2, \mathbf{r}_1, t_1)$ = probability density (=probability per unit volume) that if a particle at rest is put into the system at point $\mathbf{r}_1$ at time t_1, then it will be found at $\mathbf{r}_2$ at later time t_2. (2.7)

It will be convenient, when we later take the Fourier transform, to have P defined also for $t_2<t_1$:

$$P(\mathbf{r}_2, t_2, \mathbf{r}_1, t_1) = 0, \quad \text{for } t_2<t_1. \tag{2.8}$$

In Fig. 2.2 is a graph showing a qualitative picture of this propagator in the interacting and non-interacting cases. Probability density is plotted on the

vertical axis, and t_2 and an arbitrary component of $\mathbf{r}_2$ on the horizontal axes. In the absence of interactions, P will be a surface which is zero everywhere except on the line $\mathbf{r}_2-\mathbf{r}_1=\frac{1}{2}(\mathbf{F}/m)(t_2-t_1)^2$, where it equals ∞, i.e., the Dirac δ-function:

$$P_0(\mathbf{r}_2, t_2, \mathbf{r}_1, t_1) = \delta\left[(\mathbf{r}_2-\mathbf{r}_1)-\frac{1}{2}\left(\frac{\mathbf{F}}{m}\right)(t_2-t_1)^2\right]. \tag{2.9}$$

This propagator in the absence of interactions is called the *free propagator*.

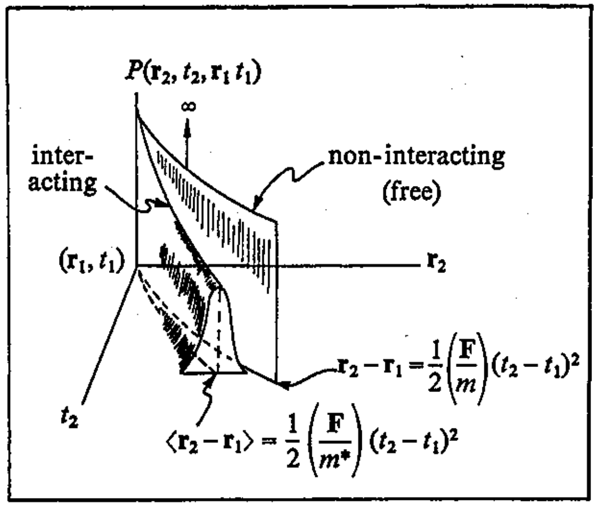


Fig. 2.2 *The Classical Propagator (Schematic—Only One Component of* $\mathbf{r}_2$ *Shown)*

If interactions between particles are now allowed to occur, this surface will spread out, as shown qualitatively. If we examine $\langle\mathbf{r}_2-\mathbf{r}_1\rangle$, the position of the maximum value of P in the interacting case, we see that for some types of interaction we might find that

$$\langle\mathbf{r}_2-\mathbf{r}_1\rangle = \frac{1}{2}\left(\frac{\mathbf{F}}{m^*}\right)(t_2-t_1)^2 \quad \text{for } P = \text{maximum}. \tag{2.10}$$

If this is true, then $\langle\mathbf{r}_2-\mathbf{r}_1\rangle$ behaves as the co-ordinate of a quasi particle of effective mass m^*. Look now at the maximum height of P as a function of t_2. Because of the 'spreading out' of the particle position, P_{max} will first fall infinitely rapidly from its value of ∞ at $t_2=t_1$, then more slowly. If this slower decay is exponential:

$$P_{max}(\mathbf{r}_2, t_2, \mathbf{r}_1, t_1) \propto e^{-(t_2-t_1)/\tau}, \tag{2.11}$$

then τ may be identified as the quasi particle lifetime; it clearly must be fairly large if the quasi particle picture is to be useful. Thus, if we calculate P and find that it shows the above behaviour, then the system is describable in terms of quasi particles and their lifetime and effective mass may be determined.

2.3 Calculation of the propagator by means of diagrams

The actual calculation of the propagator P is quite complicated, but it is easy to illustrate all the principles involved with the aid of a simple analogue example in which the many-body system is replaced by a set of fixed scattering centres. (The system considered here is essentially the same as the drunken man case in chapter 1, but it will be treated in much more detail.)

The example involves the particle accelerator in Fig. 2.3. A pinball is injected at the point $\mathbf{r}_1$, at time t_1 and propagates through the system, being scattered at the various centres. We ask for the probability $P(\mathbf{r}_2, t_2, \mathbf{r}_1, t_1)$ that the particle reaches the point $\mathbf{r}_2$ at time t_2.

The scattering mechanism is assumed to be such that (1) if the pinball strikes the shaded circle at animal A, then there is probability $P(A)$ that it is scattered and $1-P(A)$ that it will go straight through without scattering, (2) the probability distribution of pinball paths and velocities after scattering at A must be independent of the pinball path and velocity before scattering—that is, the pinball loses its 'memory' of how it got to A.

(There are many ways in which the above properties can be approximately realized. For example, the shaded circle could be a round peg which is pushed up so that it protrudes above the playing board surface a fraction $P(A)$ of the time, and is pulled in so that it is flush with the surface (hence cannot scatter the pinball) the rest of the time. Or we could have an immovable peg (i.e., always protruding) within the shaded circle, having a diameter such that the ratio of the peg diameter to that of the circle $= P(A)$. The loss of memory could be achieved by attaching a 'shuffling' device to each peg—like for example rapidly rotating spokes. The choice of method and the 'Rube Goldberg' details are, however, left as an exercise to the reader. They are of no importance for our discussion!)

For the sake of simplicity, let us leave time out of the argument to begin with, and consider just $P(\mathbf{r}_2, \mathbf{r}_1)$; this is the probability that if the particle begins at $\mathbf{r}_2$ it will finish at $\mathbf{r}_2$ regardless of the time. From the definition of probability, $P(\mathbf{r}_2, \mathbf{r}_1)$ is the sum of the probabilities for all the different ways the particle can go through the machine which begin at $\mathbf{r}_1$ and wind up at $\mathbf{r}_2$. For example, it could go 'directly' from $\mathbf{r}_1$ to $\mathbf{r}_2$ (i.e., without being scattered on the way) or it could go from $\mathbf{r}_1$ to the giraffe, be scattered off the giraffe and fall to $\mathbf{r}_2$. Or it could scatter from the giraffe to the monkey to $\mathbf{r}_2$. Or it could scatter twice on the giraffe before falling to $\mathbf{r}_2$. And so on.

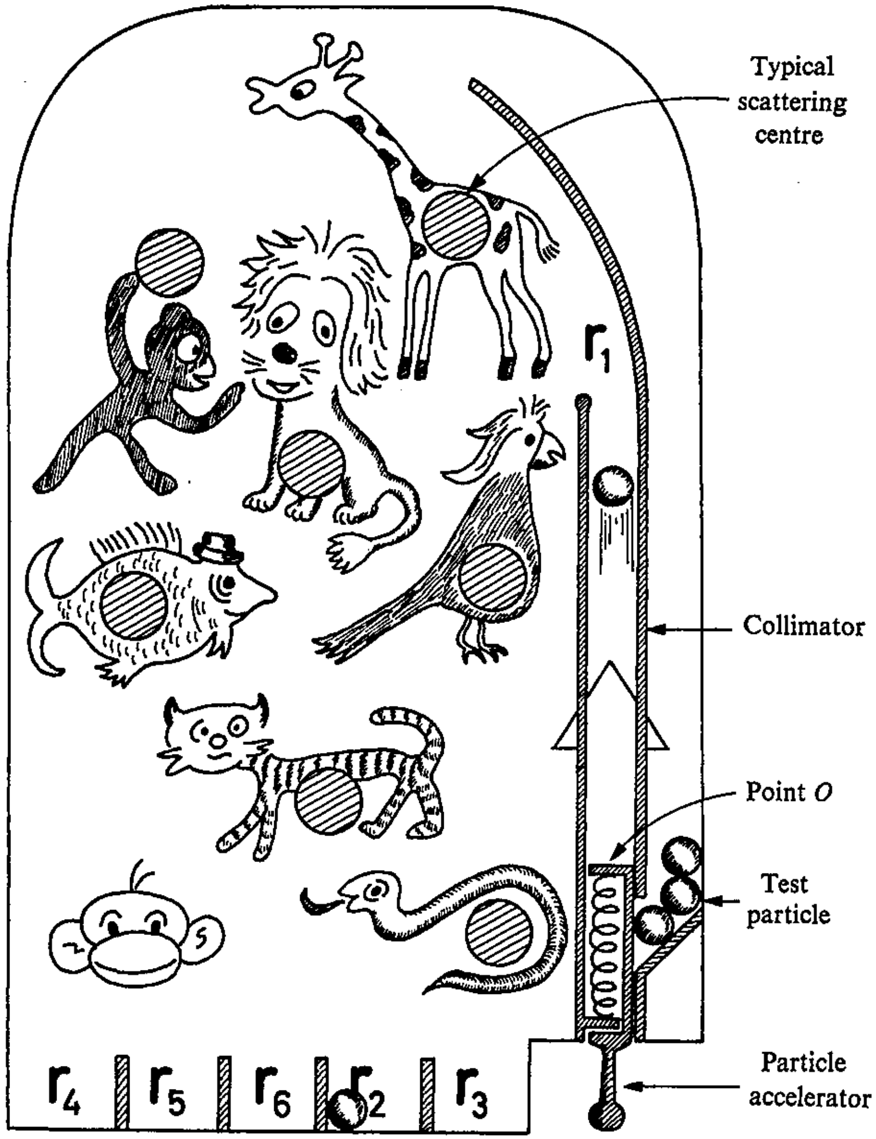


Fig. 2.3 *Classical Analogue Machine to Illustrate the Single-particle Propagator*

We first calculate the probability that the pinball will follow any particular path through the system. Let $P_0(\mathbf{r}_j,\mathbf{r}_i)$ = probability that if the pinball leaves $\mathbf{r}_i$ then it travels to $\mathbf{r}_j$ without being scattered by an animal en route ('free propagator'). The simplest path the pinball can follow is from $\mathbf{r}_1$ to $\mathbf{r}_2$ without scattering; this has probability $P_0(\mathbf{r}_2,\mathbf{r}_1)$. Another path is from $\mathbf{r}_1$ to the giraffe at $\mathbf{r}_G$ (probability $=P_0(\mathbf{r}_G,\mathbf{r}_1)$), scattering at the giraffe (probability $=P(G)$), then from $\mathbf{r}_G$ to $\mathbf{r}_2$ (probability $=P_0(\mathbf{r}_2,\mathbf{r}_G)$). Because the pinball loses its memory after the scattering at $\mathbf{r}_G$, these probabilities are independent of each other, and the joint probability for the whole path is just the product of the probabilities for each part of the path:

$$P\{(\mathbf{r}_1 \to \mathbf{r}_G), (\text{scattered at } \mathbf{r}_G), (\mathbf{r}_G \to \mathbf{r}_2)\} = P_0(\mathbf{r}_G,\mathbf{r}_1)P(G)P_0(\mathbf{r}_2,\mathbf{r}_G). \quad (2.12)$$

(Note that a process in which the pinball goes from $\mathbf{r}_1$ to $\mathbf{r}_G$, is not scattered at $\mathbf{r}_G$, and continues to $\mathbf{r}_2$, is not included in (2.12), but in the free propagator, $P_0(\mathbf{r}_2,\mathbf{r}_1)$.) The probabilities for the other paths are calculated in a similar fashion.

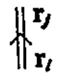
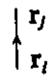
The total probability, $P(\mathbf{r}_2,\mathbf{r}_1)$, is just the sum of the probabilities for the various paths. Thus we find

$$P(\mathbf{r}_2,\mathbf{r}_1) = P_0(\mathbf{r}_2,\mathbf{r}_1)+P_0(\mathbf{r}_G,\mathbf{r}_1)P(G)P_0(\mathbf{r}_2,\mathbf{r}_G)+P_0(\mathbf{r}_M,\mathbf{r}_1)P(M)P_0(\mathbf{r}_2,\mathbf{r}_M)+$$
$$+P_0(\mathbf{r}_G,\mathbf{r}_1)P(G)P_0(\mathbf{r}_G,\mathbf{r}_G)P(G)P_0(\mathbf{r}_2,\mathbf{r}_G)+\cdots \quad (2.13)$$

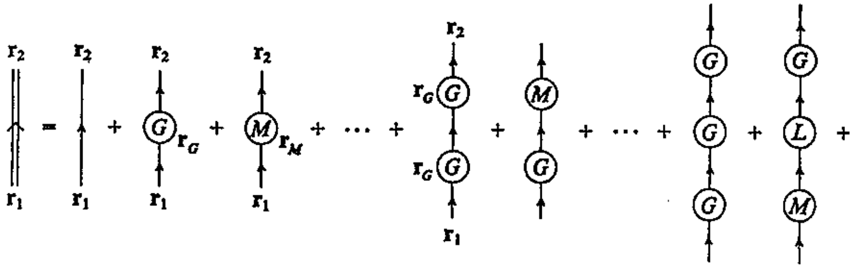
where G = giraffe, M = monkey, etc. What we have here is evidently just a perturbation expansion of the propagator, in which the $P(A)$'s play the same sort of role that the matrix elements of the perturbation, V_{kl}, play in quantum mechanical perturbation expansions.

In order to make series (2.13) easier to interpret, we draw a 'picture dictionary' to associate diagrams with the various probabilities as shown in Table 2.1.

Table 2.1 *Diagram dictionary for the pinball propagator*

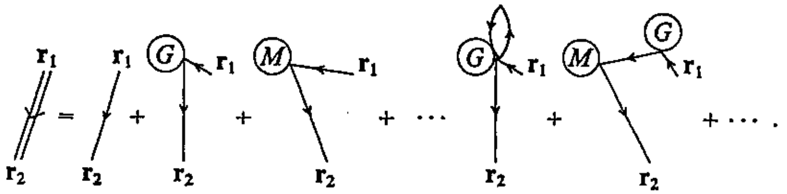
Word	Picture
$P(\mathbf{r}_j,\mathbf{r}_i)$	
$P_0(\mathbf{r}_j,\mathbf{r}_i)$	
$P(A)$	

Then the series (2.13) may be drawn thus:



(2.14)

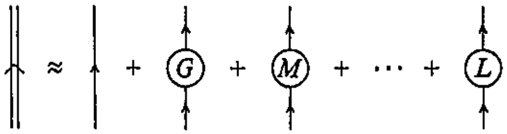
Equations (2.14) and (2.13) are of course completely equivalent to each other, being in one-to-one correspondence by the dictionary Table 2.1. But the picture has the advantage of revealing the physical meaning of (2.13), showing directly the particle shooting out from $\mathbf{r}_1$, undergoing various sequences of collisions and coming finally to $\mathbf{r}_2$. It presents in a vivid and systematic way the total probability as the sum of the probabilities associated with all the possible paths or 'histories' the particle can have as it goes through the system. Note that it is possible to interpret the $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_G$, ... on the diagrams as being points in real space if we just re-draw the diagrams so the points lie as in Fig. 2.3 thus:



(2.15)

It is important to observe that in terms of diagrams, 'the sum of the probabilities for all the different ways the particle can go from $\mathbf{r}_1$ to $\mathbf{r}_2$, interacting with the various scatterers' may be translated into 'the sum of all possible different diagrams which can be built up out of labelled circles connected by directed lines, beginning at $\mathbf{r}_1$ and terminating at $\mathbf{r}_2$'. This is because there is just one diagram corresponding to each physical path through the system.

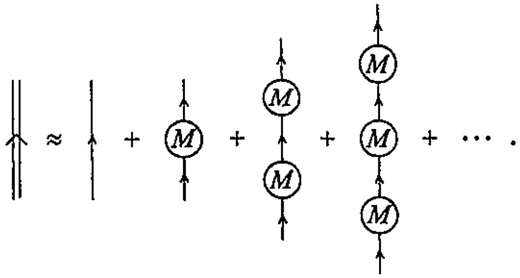
How can this series be evaluated? If we assume that the P_0's are large, say $\sim\frac{1}{2}$ or so, and the various interaction $P(A)$'s are small, say $\sim\frac{1}{10}$, then the higher order diagrams (i.e., terms; note that by order here we mean the total number of interactions) will give successively smaller contributions, and just as in ordinary perturbation theory, we can get an approximate solution by simply summing the series up through the first- or second-order terms. Thus, the zeroth-order approximation would be just the unperturbed case where the particle propagates freely from $\mathbf{r}_1$ to $\mathbf{r}_2$. When we add the possibility of a perturbing (scattering) interaction with the various animals just once each, we get the first-order approximation



(2.16)

Allowing two interactions gives the second-order approximation and so on. If, on the other hand, one or more of the interaction terms $P(A)$ is large (i.e., strong scattering at A) this method is not practical, since the series converges too slowly, and the summation must be carried out to extremely high orders to give a good result.

However, there is another kind of approximation we can make in this strong interaction case, an approximation that does not stop at second order, but instead sums over diagrams to infinite order. Suppose, for example, that only P (monkey) is large and all the other $P(A)$'s are small. Then the monkey diagrams will dominate, and the series may be approximated by the sum over just repeated monkeys, thus:



(2.17)

Translating each element of the diagrams into the appropriate probability, it is easy to write down the corresponding series:

$$P(\mathbf{r}_2,\mathbf{r}_1) \approx P_0(\mathbf{r}_2,\mathbf{r}_1)+P_0(\mathbf{r}_M,\mathbf{r}_1)P(M)P_0(\mathbf{r}_2,\mathbf{r}_M)+ \\ +P_0(\mathbf{r}_M,\mathbf{r}_1)P(M)P_0(\mathbf{r}_M,\mathbf{r}_M)P(M)P_0(\mathbf{r}_2,\mathbf{r}_M)+\cdots. \quad (2.18)$$

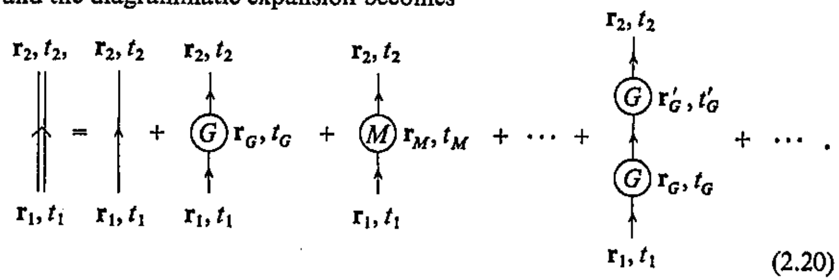
And now we notice that this infinite series is easily summed, since it is just a geometric progression:

$$P(\mathbf{r}_2,\mathbf{r}_1) \approx P_0(\mathbf{r}_2,\mathbf{r}_1)+P_0(\mathbf{r}_M,\mathbf{r}_1)P(M)P_0(\mathbf{r}_2,\mathbf{r}_M)\times \\ \times[1+P(M)P_0(\mathbf{r}_M,\mathbf{r}_M)+P(M)^2P_0(\mathbf{r}_M,\mathbf{r}_M)^2+\cdots] \\ = P_0(\mathbf{r}_2,\mathbf{r}_1)+\frac{P_0(\mathbf{r}_M,\mathbf{r}_1)P(M)P_0(\mathbf{r}_2,\mathbf{r}_M)}{1-P(M)P_0(\mathbf{r}_M,\mathbf{r}_M)}. \quad (2.19)$$

Thus, we have obtained an approximate solution for the propagator $P(\mathbf{r}_2,\mathbf{r}_1)$ which is valid in the strong interaction case.

This new approximation, involving the summation of a perturbation series to infinite order over a selected class of repeated diagrams (i.e., terms) is called '*partial summation*' or '*selective summation*'. It is drastically different from the ordinary perturbation approximation. It goes beyond conventional perturbation theory and can be used in cases where the interaction term is so large that the ordinary low-order perturbation approximation won't work. It is this property which makes the new technique of great value in tackling the strong interactions encountered in the many-body problem. As will be seen shortly, this method of partial summation is the basic procedure underlying the calculation of the quantum mechanical propagator.

The above diagram technique may easily be extended to the time-dependent propagator, $P(\mathbf{r}_2,\mathbf{r}_1,t_2-t_1)$. (We have written t_2-t_1 since the force is time independent so the propagators can depend only on time differences.) Let $P_0(\mathbf{r}_j,\mathbf{r}_i,t_j-t_i)$ = probability that if the particle leaves the point $\mathbf{r}_i$ at time t_i then it arrives at $\mathbf{r}_j$ at time t_j without undergoing any interaction on the way (this is the 'free propagator'). Let $P(A)$ be the interaction term, assumed instantaneous for simplicity. Then, using the convention that time increases in the positive y direction, the new diagram dictionary is given by Table 2.2 and the diagrammatic expansion becomes



(2.20)

(Analogous to (2.15), these diagrams may be re-drawn (at least in the one-dimensional case) in a co-ordinate system with t as ordinate, and r as abscissa.) Then, in writing down the corresponding series, it is necessary to remember that t_A, the time at which the scattering from A occurs, may be anywhere between t_1 and t_2, and there is some probability that it occurs at any of these intermediate moments. Thus, the total probability is the sum of all these contributions, and this implies that we must integrate over all intermediate times, t_A. This leads to the series:

$$P(\mathbf{r}_2,\mathbf{r}_1,t_2-t_1) = P_0(\mathbf{r}_2,\mathbf{r}_1,t_2-t_1)+ \\ +\int_{t_1}^{t_2} dt_G P_0(\mathbf{r}_G,\mathbf{r}_1,t_G-t_1)P(G)P_0(\mathbf{r}_2,\mathbf{r}_G,t_2-t_G)+ \\ +\int dt_M\cdots+\int\int+\cdots+\int\int\int+\cdots+\cdots. \quad (2.21)$$

Table 2.2 *Dictionary for pinball propagator including time*

Word	Diagram
$P(\mathbf{r}_j,\mathbf{r}_i,t_j-t_i)$	$\mathbf{r}_j, t_j$ ⇑ $\mathbf{r}_i, t_i$
$P_0(\mathbf{r}_j,\mathbf{r}_i,t_j-t_i)$	$\mathbf{r}_j, t_j$ ↑ $\mathbf{r}_i, t_i$
$P(A)$	Ⓐ

The unpleasant integrals parading through this expression may be removed by noticing that they all have the form of 'folded' products. This means they can be converted into simple products by a Fourier transformation. Suppose we define the transformed propagator, $P_0(\mathbf{r}_j,\mathbf{r}_i,\omega)$ (ω = frequency) by

$$P_0(\mathbf{r}_j,\mathbf{r}_i,t_j-t_i) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega(t_j-t_i)} P_0(\mathbf{r}_j,\mathbf{r}_i,\omega) \tag{2.22}$$

with a similar expression for $P(\mathbf{r}_j,\mathbf{r}_i,\omega)$. Then the first two terms of (2.21) become (note that we can integrate over t_G from $-\infty$ to $+\infty$ because condition (2.8) automatically limits the integral to the region $t_1 \to t_2$):

$$P_0(\mathbf{r}_2,\mathbf{r}_1,t_2-t_1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega(t_2-t_1)} P_0(\mathbf{r}_2,\mathbf{r}_1,\omega)$$

$$\int_{-\infty}^{+\infty} dt_G\, P_0(\mathbf{r}_G,\mathbf{r}_1,t_G-t_1)P(G)\,P_0(\mathbf{r}_2,\mathbf{r}_G,t_2-t_G) =$$

$$= \int_{-\infty}^{+\infty} dt_G \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega'\, e^{-i\omega'(t_G-t_1)} P_0(\mathbf{r}_G,\mathbf{r}_1,\omega')\right] \times$$

$$\times P(G) \times \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega(t_2-t_G)} P_0(\mathbf{r}_2,\mathbf{r}_G,\omega)\right] =$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega'\, P_0(\mathbf{r}_G,\mathbf{r}_1,\omega') \times$$

$$\times P(G)P_0(\mathbf{r}_2,\mathbf{r}_G,\omega)\, e^{+i(\omega' t_1-\omega t_2)} \underbrace{\int_{-\infty}^{+\infty} dt_G\, e^{-it_G(\omega'-\omega)}}_{2\pi\delta(\omega'-\omega)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega(t_2-t_1)} P_0(\mathbf{r}_G,\mathbf{r}_1,\omega)P(G)P_0(\mathbf{r}_2,\mathbf{r}_G,\omega). \tag{2.23}$$

Continuing thus, and finally taking the inverse transform, yields

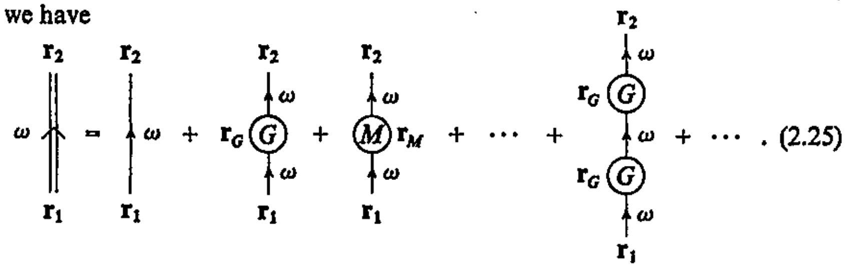
$$P(\mathbf{r}_2,\mathbf{r}_1,\omega) = P_0(\mathbf{r}_2,\mathbf{r}_1,\omega)+P_0(\mathbf{r}_G,\mathbf{r}_1,\omega)P(G)P_0(\mathbf{r}_2,\mathbf{r}_G,\omega)+\cdots. \tag{2.24}$$

This is just as simple as the series (2.13) for the time-independent case. We can use the partial summation trick on it just as before. Thus inclusion of time in the propagator creates no special difficulties. Note that the Fourier transformed series may be gotten directly by using a revised edition of the 'dictionary', Table 2.2, in which the diagrams are for transforms of the propagators, as in Table 2.3. Hence the diagrams for (2.24) are just the same

Table 2.3 *Fourier transformed pinball dictionary*

Word	Diagram
$P(\mathbf{r}_j,\mathbf{r}_i,\omega)$	ω ⇑ $\mathbf{r}_j$, $\mathbf{r}_i$
$P_0(\mathbf{r}_j,\mathbf{r}_i,\omega)$	ω ↑ $\mathbf{r}_j$, $\mathbf{r}_i$
$P(A)$	Ⓐ

as those in (2.20) provided we erase all the t's and put in all the ω's. Thus, we have



We shall not actually apply this formalism to the calculation of classical quasi particles—this would take us too far afield. Instead we go on directly to the quantum case.

Exercises

2.1 Write the diagram series for the propagator $P(\mathbf{r}_2,\mathbf{r}_1)$ assuming that the scattering at both the monkey and the lion are large, while all other interactions are small. Include all terms through second order, plus a couple of third-order terms. How many diagrams are there in nth order?
2.2 Translate the first few terms of Ex. 2.1 into functions.
2.3 Evaluate the above propagator by partial summation assuming that all $P_0(\mathbf{r}_i,\mathbf{r}_j)=c$.
2.4 Assuming all free propagators $=c$, generalize the above results to include scattering from all animals.

Chapter 3

Quantum Quasi Particles and the Quantum Pinball Propagator

3.1 The quantum mechanical propagator

In this chapter we are going to solve the simplest existing example of a quantum field theoretical problem. We call it the 'quantum pinball game' since it is the precise quantum analogue of the classical pinball machine just discussed, and in fact gives rise to a diagrammatic series having exactly the same form as (2.25). It is a sub-trivial problem, one which can be solved in a microsecond by elementary quantum mechanics. It takes a little longer to do by diagrams, but like its classical cousin in Fig. 2.3 has the great merit of illustrating all the basic principles without immersing the reader in a morass of mathematics. At the end of the chapter, the diagram method is applied to a non-trivial problem, i.e., finding the energy and lifetime of an electron propagating through a set of randomly distributed scattering centres (e.g., impurity atoms in a metal).

The fundamental difference between the classical propagator, P, and the quantum propagator, G, is that P is a probability, whereas G is a probability *amplitude*, with corresponding probability given by $|G|^2$ $(= G^* G)$. Thus in the classical case, the total probability for propagation from point 1 to point 2 is just the sum of the probabilities for each propagation process taken separately:

$$P(2,1)_{\text{classical}} = P(\text{process I}) + P(\text{process II}) + \cdots.$$

But in the quantum case, the total probability *amplitude* is the sum of the probability *amplitudes* for each process taken separately

$$G(2,1) = G(\text{process I}) + G(\text{process II}) + \cdots$$

so that the corresponding probability is given by

$$P(2,1)_{\text{quantum}} = G^* G = \underbrace{|G(\text{I})|^2}_{P(\text{I})} + \underbrace{|G(\text{II})|^2}_{P(\text{II})} + \underbrace{G(\text{I})^* G(\text{II}) + G(\text{II})^* G(\text{I})}_{\text{interference terms}} + \cdots.$$

Because of the characteristic 'interference terms', the quantum probability is not just the sum of the probabilities for the individual processes, in contrast to the classical case.