Multicritical point, conformal field theory and duality

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Introduction

Critical phenomena are one of the important subjects in condensed matter physics. Many developments about critical phenomena, such as renormalization group, numerical methods etc. have been done. But, when the model has a multicritical point, the scaling behaviors become difficult due to the interference of multiple critical lines. So, conventional numerical methods are not useful near a multicritical point.
We have studied several multicritical phenomena combining with the conformal field theory (CFT) and numerical methods (level spectroscopy etc). And we discuss the relation with the duality, such as the Kramers-Wannier duality and the Ashkin-Teller self-duality.

Introduction

Topics about multicritical phenomena:

1. bifurcation at c=3/2 CFT to the two c=1 CFT lines (Takhtajan-Babujian)
2. bifurcation of c=1 CFT to the two c=1/2 CFT lines (Ashkin-Teller)
3. crossing of c=1 CFT (TLL) with c=1/2 CFT (Schulz multicritical point)

S=1 BLBQ chain with bond-alternation

A. Kitazawa and K. N. : Phys. Rev. B **59**,pp.11358 (1999)
1D S=1 Bilinear-Biquadratic model with bond-alternation

$$\hat{H} = \sum_{j=1}^{L} (1 - (-1)^j \delta) \left(\cos\theta \hat{\boldsymbol{S}}_j \hat{\boldsymbol{S}}_{j+1} + \sin\theta (\hat{\boldsymbol{S}}_j \hat{\boldsymbol{S}}_{j+1})^2 \right) \tag{1}$$

1. $\theta = -\pi/4$ and $\delta = 0$: Takhtajan-Babujian (TB) point
   Bethe Ansatz solvable, massless, c=3/2 Conformal field theory (CFT) or SU(2) level 2 Wess-Zumino-Witten (WZW) model
2. $\theta < -\pi/4$ and $\delta = 0$:
   dimer order, two-fold degenerate
3. $\theta > -\pi/4$:
   there are two c=1 CFT critical lines

S=1 BLBQ chain with bond-alternation

Phase diagram of 1D S=1 BLBQ with bond-alternation

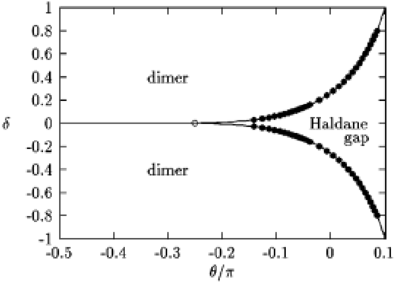


FIG. 1. Phase diagram of the S=1 spin chains (1). TB point is $\theta = -\pi/4$ and $\delta = 0$ (○), which is the multicritical point. Lines in the region $-\pi/4 < \theta < \arctan(1/3)$ are $c=1$ critical lines. ● is the extrapolated numerical data for $N = 8,10,12,14,16$ systems (Sec. III). The line $\delta = 0 \theta < -\theta/4$ is a first-order phase-transition line.[4,10]

Figure: Phase diagram

S=1 BLBQ chain with bond-alternation

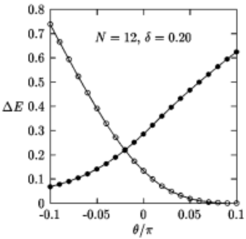


FIG. 3. Level crossing in the system with the twisted boundary conditions. ● is the lowest P = T = 1 energy $E_e - E_g$, and ○ is the lowest P = T = −1 energy $E_o - E_g$ for the N = 12 systems. We subtract the ground-state energy with the periodic boundary condition.

Figure: energies of TBC

Twisted boundary condition(TBC)

$$\hat{S}^x_{L+j} = -\hat{S}^x_j, \hat{S}^y_{L+j} = -\hat{S}^y_j, \hat{S}^z_{L+j} = \hat{S}^z_j. \quad (2)$$

A. Kitazawa: J. Phys. A **30**, L285 (1997)

S=1 BLBQ chain with bond-alternation

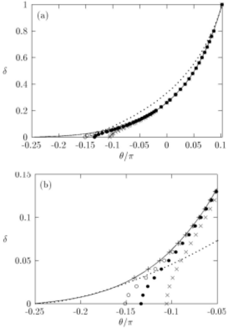


FIG. 4. Crossing points for N=8 (×), N=12 (●), and N = 16 (○) systems. +'s are the extrapolated data, and the solid line is the interpolated one of it. Dotted lines are (a) Eq. (27) and (b) $\delta=(\theta/\pi+0.25)^{13/8}$.

Figure: TBC crossing lines at various sizes

relevant scaling dimensions: $x = 3/8$ and $x = 1$

S=1 BLBQ chain with bond-alternation

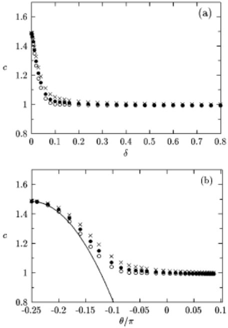


FIG. 6. Conformal anomaly c(N) on the c=1 critical line for N=8 (×), N=10 (●), and N=12 (○) systems, as a function of (a) δ_c and (b) θ_c. In (b), we also show the line $c_{TB}-\alpha(\theta+0.25\pi)^2$, in which numerical number c_{TB} and α are determined by data at $\theta=-0.25\pi$ and -0.24π for c(N=12).

Figure: Effective central charge

$$\frac{E_g(L)}{L} = e_g - \frac{\pi v c}{6L^2} \tag{3}$$

Ashkin-Teller and S=1/2 bond-alternating XXZ chain

M. den Nijs, M. Kohmoto and L. P. Kadanoff: Phys. Rev. B **24** pp. 5229 (1981)

2D classical Ashkin-Teller model

$$H = \sum_{i,j} (K_1 S_i S_j + K_2 T_i T_j + K_3 S_i S_j T_i T_j), \quad (S_i = \pm 1, T_i = \pm 1) \tag{4}$$

→ 1D quantum Ashkin-Teller model (transfer matrix of the 2D classical Ashkin-Teller model)

$$\hat{H} = \sum_{j}^{L} (\hat{\sigma}_j^z \hat{\sigma}_{j+1}^z + \hat{\tau}_j^z \hat{\tau}_{j+1}^z + \lambda \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z \hat{\tau}_j^z \hat{\tau}_{j+1}^z) - \sum_{j}^{L} (\hat{\sigma}_j^x + \hat{\tau}_j^x + \lambda \hat{\sigma}_j^x \hat{\tau}_j^x) \tag{5}$$

($\hat{\sigma}, \hat{\tau}$: Pauli matrices)

→ S=1/2 bond-alternating chain

$$\hat{H} = \sum_{j=1}^{L} (1 - (-1)^j \delta) \left(\hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \Delta \hat{S}_j^z \hat{S}_{j+1}^z \right) \tag{6}$$

Ashkin-Teller and S=1/2 bond-alternating XXZ chain

S=1/2 bond-alternating chain

$$\hat{H} = \sum_{j=1}^{L} (1 - (-1)^j \delta) \left(\hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \Delta \hat{S}_j^z \hat{S}_{j+1}^z \right)$$

- Berezinskii-Kosterlitz-Thouless (BKT) transition on $\delta = 0$ line
- scaling dimensions at $\delta = 0, \Delta = 1$ point: $x = 2$ and $x = 1/2$
- numerically difficult to caluculate multicritical lines.

S=1/2 bond-alternating XXZ chain

S. Moriya and K.N.: J.P.S.J. Vol. 89,093001 (2020)

Phase diagram of S=1/2 bond-alternating chain

$$\hat{H} = \sum_{j=1}^{L} (1 - (-1)^j \delta) \left(\hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \Delta \hat{S}_j^z \hat{S}_{j+1}^z \right)$$

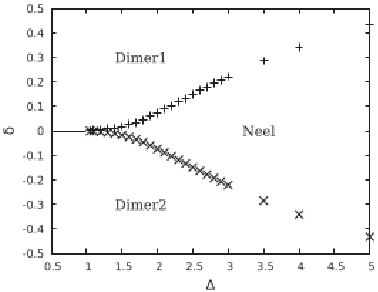


Fig. 1. Phase diagram in the Δ–δ plane. Dimer1–Dimer2 phase boundary is the Gaussian universality and Dimer–Néel phase boundaries are the 2D Ising universality. We draw the 2D Ising universality transition lines by using the $L = 24$ numerical result of yTBC–zTBC method, noted by $+$ and $\times$.

S=1/2 bond-alternating XXZ chain

$$\hat{H} = \beta \sum_{j=1}^{L/2} \left(\hat{S}^x_{2j} \hat{S}^x_{2j+1} + \hat{S}^y_{2j} \hat{S}^y_{2j+1} + \Delta \hat{S}^z_{2j} \hat{S}^z_{2j+1} \right) + \sum_{j=1}^{L/2} \left(\hat{S}^x_{2j-1} \hat{S}^x_{2j} + \hat{S}^y_{2j-1} \hat{S}^y_{2j} + \Delta \hat{S}^z_{2j-1} \hat{S}^z_{2j} \right) \quad (7)$$

$(\beta = (1-\delta)/(1+\delta))$
In $\Delta \to \infty, \beta \to 0, \Delta\beta = O(1)$ limit, the main contribution of the Hamiltonian is the $\Delta \hat{S}^z_{2j-1} \hat{S}^z_{2j}$ term.

$$|\uparrow_{2j-1}, \downarrow_{2j}\rangle = |\uparrow'_j\rangle \qquad |\downarrow_{2j-1}, \uparrow_{2j}\rangle = |\downarrow'_j\rangle \quad (8)$$

Pertubative Hamiltonian:

$$\hat{H}_1 = \beta\Delta \sum_{j=1}^{L/2} \hat{S}^z_{2j} \hat{S}^z_{2j+1} + \sum_{j=1}^{L/2} \left(\hat{S}^x_{2j-1} \hat{S}^x_{2j} + \hat{S}^y_{2j-1} \hat{S}^y_{2j} \right) = \beta\Delta \sum_{j=1}^{L/2} \hat{S}^z_{2j} \hat{S}^z_{2j+1} + \frac{1}{2} \sum_{j=1}^{L/2} \left(\hat{S}^+_{2j-1} \hat{S}^-_{2j} + \hat{S}^-_{2j-1} \hat{S}^+_{2j} \right) \quad (9)$$

S=1/2 bond-alternating XXZ chain

$$
\begin{aligned}
\hat{S}^z_{2j+1}|\uparrow'_{j+1}\rangle &= \frac{1}{2}|\uparrow'_{j+1}\rangle \\
\hat{S}^z_{2j+1}|\downarrow'_{j+1}\rangle &= -\frac{1}{2}|\downarrow'_{j+1}\rangle \\
\hat{S}^z_{2j}|\uparrow'_{j}\rangle &= \frac{1}{2}|\uparrow'_{j}\rangle \\
\hat{S}^z_{2j}|\downarrow'_{j}\rangle &= -\frac{1}{2}|\downarrow'_{j}\rangle
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\hat{S}^+_{2j-1}\hat{S}^-_{2j}|\downarrow'_{j+1}\rangle &= |\uparrow'_{j+1}\rangle \\
\hat{S}^+_{2j-1}\hat{S}^-_{2j}|\uparrow'_{j+1}\rangle &= |\downarrow'_{j+1}\rangle
\end{aligned}
\tag{11}
$$

S=1/2 bond-alternating XXZ chain

In summary, the effective Hamiltonian:

$$\hat{H}' = \sum_{j=1}^{L/2} \left(-\beta\Delta \hat{S}'^{z}_{j} \hat{S}'^{z}_{j+1} + \hat{S}'^{x}_{j} \right) \tag{12}$$

By operating $\exp(i\pi \sum \hat{S}'^{z}_{j})$, this is equivalent with the Transverse Field Ising (TFI) model:

$$\hat{H}' = \beta\Delta \sum_{j=1}^{L/2} \left(-\hat{S}'^{z}_{j} \hat{S}'^{z}_{j+1} - \gamma \hat{S}'^{x}_{j} \right), \quad (\gamma \equiv \frac{1}{\beta\Delta}) \tag{13}$$

(TFI model comes from the transfer matrix of the classical 2D Ising model).
Note that $\gamma = 1$ is a critical point with 2D Ising type.

TFI model, Kramers-Wannier duality and boundary condition

$$\hat{H}' = -\sum_{j=1}^{L/2-1} \hat{S}_j'^z \hat{S}_{j+1}'^z - g\hat{S}_{L/2}'^z \hat{S}_1'^z - \gamma \sum_{j=1}^{L/2} \hat{S}_j'^x, \tag{14}$$

($g = 1$:periodic boundary condition (PBC), $g = -1$:antiperiodic boundary condition (ABC)).
From duality (using Jordan-Wigner type transformation), one obtain

$$E_0(L, g = 1, U_\pi^y = -1) = E_0(L, g = -1, U_\pi^y = 1) + 2(\gamma - 1) \tag{15}$$

Thus one can determine the critical point with the crossing

$$E_0(L, g = 1, U_\pi^y = -1) = E_0(L, g = -1, U_\pi^y = 1) \tag{16}$$

1D BA XXZ model and BC

z-axis twisted boundary condition (zTBC):

$$\hat{S}^x_{L+j} = -\hat{S}^x_j, \hat{S}^y_{L+j} = -\hat{S}^y_j, \hat{S}^z_{L+j} = \hat{S}^z_j. \quad (17)$$

y-axis BC (yTBC):

$$\hat{S}^x_{L+j} = -\hat{S}^x_j, \hat{S}^y_{L+j} = \hat{S}^y_j, \hat{S}^z_{L+j} = -\hat{S}^z_j. \quad (18)$$

1. PBC,zTBC of the 1D BA XXZ $\leftrightarrow$ $g = 1$ of the 1D TFI
2. yTBC of the 1D BA XXZ $\leftrightarrow$ $g = -1$ of the 1D TFI

Thus, in 1D BA XXZ,

$$E_0^{PBC}(M = 0, U^y_\pi = -1) = E_0^{yTBC}(M = even, U^y_\pi = 1) \quad (19)$$

or

$$E_0^{zTBC}(M = 0, U^y_\pi = -1) = E_0^{yTBC}(M = even, U^y_\pi = 1) \quad (20)$$

1D BA XXZ model and BC

On $\delta = 0$ we can show

$$E_0^{zTBC}(L, M = 0, U_\pi^y = -1) = E_0^{zTBC}(L, M = 0, U_\pi^y = 1) \qquad (21)$$

Thus, at $\delta = 0$ and $\Delta = 1$ (isotropic) point,

$$E_0^{zTBC}(M = 0, U_\pi^y = -1) = E_0^{yTBC}(M = even, U_\pi^y = 1) \qquad (22)$$

S=1/2 bond-alternating XXZ chain

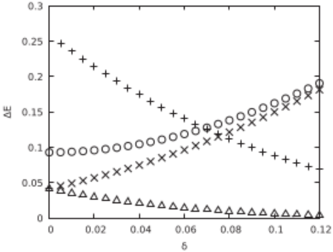


Fig. 2. The energies of each BC for $L = 14$. The value of Δ is fixed at 2.0 and δ is changed. $\bigcirc$ is $E_0^{\mathrm{PBC}}(M=0, U_\pi^y=-1)$, $\times$ is $E_0^{z\mathrm{TBC}}(M=0, U_\pi^y=-1)$, Δ is $E_0^{z\mathrm{TBC}}(M=0, U_\pi^y=1)$, $+$ is $E_0^{y\mathrm{TBC}}(M=\text{even}, U_\pi^y=1)$. The PBC lowest energy $E_0^{\mathrm{PBC}}(M=0, U_\pi^y=1)$ is subtracted from each energy.

Figure: energies of each BC

S=1/2 bond-alternating XXZ chain

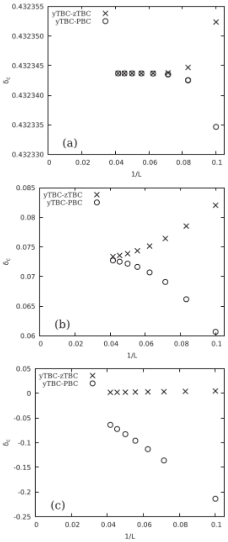


Fig. 3. The size dependence of the crossing point δ_c of the yTBC–PBC method (○) and the yTBC–zTBC method (×), (a) in $\Delta = 5.0$, (b) in $\Delta = 2.0$, (c) in $\Delta = 1.1$.

Figure: Comparison between yTBC-PBC and yTBC-zTBC

S=1/2 bond-alternating XXZ chain

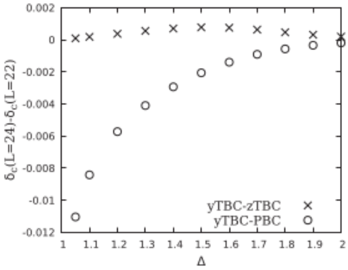


Fig. 4. The size difference of the crossing point δ_c $L = 24$ and 22. O is the yTBC–PBC method, × is the yTBC–zTBC method.

Figure: Comparison between yTBC-PBC and yTBC-zTBC

S=1 XXZ with single ion anisotropy

S=1 XXZ + single ion anisotropy chain

$$H = \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z + D(S_j^z)^2 \right) \tag{23}$$

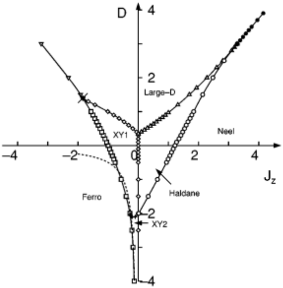


FIG. 1. The phase diagram of S=1 XXZ chains with uniaxial single-ion-type anisotropy. The solid lines and symbols are the transition lines. The dotted line shows the curve $J_z = -1/2|D|$ expected from the perturbation calculation for large negative D.

W. Chen, K. Hida and C. Sanctuary. : Phys. Rev. B **67**,pp.104401 (2003)

Bosonization by Schulz(1986)

H.J.Schulz: Phys. Rev. B Vol. 34 (1986) pp.6372

1. correlation functions in the XY1 phase

$$\langle S_j^+ S_{j+r}^- \rangle = C_\perp \exp(i\pi r)|r|^{-\eta} \tag{24}$$
$$\langle (S_j^+)^2 (S_{j+r}^-)^2 \rangle = C_{\perp 2}|r|^{-\eta_2} \tag{25}$$
$$\langle S_j^z S_{j+r}^z \rangle = C_z |r|^{-2} + D_z \exp(i\pi r)\exp(-|r|/\xi) \tag{26}$$

($\eta_2 = 4\eta$ and $0 < \eta \leq 1/4$.)

2. correlation functions in the XY2/nTLL phase

$$\langle S_j^+ S_{j+r}^- \rangle = C_\perp \exp(i\pi r)\exp(-|r|/\xi') \tag{27}$$
$$\langle (S_j^+)^2 (S_{j+r}^-)^2 \rangle = C_{\perp 2}|r|^{-\eta_2} \tag{28}$$
$$\langle S_j^z S_{j+r}^z \rangle = C_z |r|^{-2} + D_z \exp(i\pi r)|r|^{-\eta_z} \tag{29}$$

($\eta_z = 1/\eta_2$ and $0 < \eta_2 \leq 1$.)

Bosonization by Schulz(1986)

1. XY1-Haldane phase boundary: BKT transition
2. Haldane-Neel phase boundary: 2D Ising type Universality class
3. XY2/nTLL-Neel phase boundary: BKT transition
4. Haldane, Neel, XY1 and XY2/nTLL phases should cross at one point "Schulz multicritical point"

Hidden SU(2) symmetry in S=1/2 XY spin ladder system and S=1 XY spin chain

A. Kitazawa, K. Hijii and KN: J. Phys. A, Vol. 36 (2003) L351

- S=1 XY + single ion anisotropy chain

$$H = J\sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + D\sum_j (S_j^z)^2 \tag{30}$$

- S=1/2 XY quantum spin ladder system

$$\begin{aligned} H = {} & J_{leg}\sum_j (S_{1,j}^x S_{1,j+1}^x + S_{1,j}^y S_{1,j+1}^y + S_{2,j}^x S_{2,j+1}^x + S_{2,j}^y S_{2,j+1}^y) \\ & + J_{dia}\sum_j (S_{1,j}^x S_{2,j+1}^x + S_{1,j}^y S_{2,j+1}^y + S_{2,j}^x S_{1,j+1}^x + S_{2,j}^y S_{1,j+1}^y) \\ & + J_{rung,xy}\sum_j (S_{1,j}^x S_{2,j}^x + S_{1,j}^y S_{2,j}^y) + J_{rung,z}\sum_j S_{1,j}^z S_{2,j}^z \end{aligned} \tag{31}$$

There is a hidden SU(2) symmetry in S=1/2 spin ladder system and S=1 XY chain.
↔ BKT transition (in the case of massless)

Hidden SU(2) symmetry

- $$\tilde{s}_j^{\pm} \equiv \frac{1}{2}(S_j^{\pm})^2, \ \tilde{s}_j^z \equiv \frac{1}{2}S_j^z \tag{32}$$
- Commutation relations
$$[\tilde{s}_j^z, \tilde{s}_k^{\pm}] = \delta_{j,k}\tilde{s}_k^{\pm} \tag{33}$$
$$[\tilde{s}_j^+, \tilde{s}_k^-] = 2\delta_{j,k}\tilde{s}_k^z \tag{34}$$
$(\because [(S_j^+)^2, (S_j^-)^2] = -8(S_j^z)^3 + 4(2S^2 + 2S - 1)S_j^z$ and $(S_j^z)^3 = S_j^z \quad (forS = 1)$)
- $\tilde{s}_j^{\pm}, \tilde{s}_j^z$ satisfy an SU(2) algebra.

Hidden SU(2) symmetry

Although the operator $\sum_j \tilde{s}_j^z$ commutates with the S=1 XY Hamiltonian, the operators $\sum_j \tilde{s}_j^\pm$ do not commutate with Hamiltonian.

- Thus, we do a following nonlocal transformation

$$s_j^\pm = \frac{1}{2}(S^\pm)^2 U_j,\ s_j^z = \frac{1}{2} S_j^z (= \tilde{s}_j^z) \tag{35}$$

  where

$$U_1 = 1, \quad U_j = \prod_{l=1}^{j-1} \left(1 - 2(S_l^z)^2\right) = \exp(i\pi \sum_{l=1}^{j-1} S_l^z) \quad (j > 1) \tag{36}$$

- The operators $s_j^\pm,\ s_j^z$ satisfy an SU(2) algebra.
- One can prove that $s_T^\pm \equiv \sum_{j=1}^L s_j^\pm$ and $s_T^z \equiv \sum_{j=1}^L s_j^z$ commutate with S=1 XY chain with open boundary condition.

Hidden SU(2) symmetry

- But S=1 XY model with the periodic boundary condition (PBC) does not commutate.
- Combining the twisted boundary condition (TBC), one can prove the hidden SU(2) symmetry in the S=1 XY chain.

$$\begin{aligned} H = & \frac{J}{2} \sum_{j}^{L-1} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \\ & + \frac{J'}{2} (S_L^+ S_1^- \exp(\mp i \frac{\pi}{2} S_T^z) + S_L^- S_1^+ \exp(\pm i \frac{\pi}{2} S_T^z)) \end{aligned} \tag{37}$$

- The energy for magnetization $M = 4n$(n: integer) under PBC is degenerate with that for magnetization $M = 4n + 2$ under TBC.
- One can also discuss the space inversion and the wave number.

Universality class of nTLL phase

central charge c=1 conformal field theory(CFT)

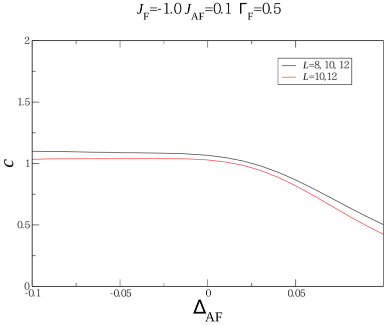


Figure: effective central charge

- Region $\Delta_{AF} \leq 0$: $c = 1$
- Region $\Delta_{AF} > 0$: $c < 1$ and c decreases as size L

Fig. 9 means that the region $\Delta_{AF} \leq 0$ belongs to the c=1 CFT universality.

Universality class of nTLL phase

Numerical results for scaling dimensions.

1. $\Delta E(M = 1, q = \pi)$: massive
2. $\Delta E(M = 2, q = 0), \Delta E(M = 4, q = 0)$: massless
3. $\Delta E(M = 4, q = 0)/\Delta E(M = 2, q = 0) = 4$,
4. $\Delta E(M = 0, q = \pi)$: massless
5. Scaling dimensions are consistent with TLL model (one parameter scaling or TL parameter K).

$$K = \sqrt{\frac{\Delta E_0(L; TBC; m = 0, P = -1)}{\Delta E_0(L; PBC; m = 2, P = 1)}} \tag{38}$$

6. TL-parameter K is consitent with the perturbative mapping (S=1/2 XXZ).

$$H = \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right), \tag{39}$$

$$K = \frac{\pi}{\arccos(-\Delta)} \tag{40}$$

They are consistent with XY2 phase by Schulz (1986).

Universality class of nTLL phase

1. Exponential quasi-degeneracy between PBC and TBC energies.

$$
\begin{aligned}
& E_0(L; TBC; m=0, P=1) - E_0(L; PBC; m=0, P=1) \\
& = C_1 \exp(-L/\xi_1) \qquad (41) \\
& E_0(L; TBC; m=2, P=1) - E_0(L; PBC; m=2, P=1) \\
& = C_2 \exp(-L/\xi_2) \qquad (42) \\
& E_0(L; TBC; m=0, P=-1) - E_0(L; PBC; m=0, P=-1) \\
& = C_3 \exp(-L/\xi_3) \qquad (43)
\end{aligned}
$$

   1.1 numerical calculation.
   1.2 perturbative mapping for S=1/2 XXZ chain.

2. The above bebavior is completely different from $\Delta(L; TBC) - \Delta(L; PBC) \propto 1/L$ in S=1/2 XXZ chain and S=1 XY1 phases.

Universality class of nTLL phase:PBC-TBC

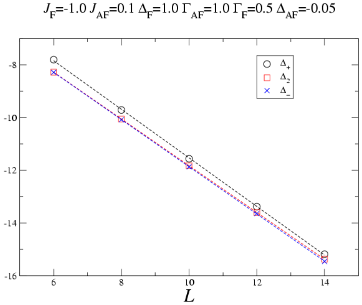


Figure: quasi degeneracy between PBC and TBC (nTLL phase), $\Gamma_F = 0.5, \Delta_{AF} = -0.05$

Exponential quasi-degenaracy (Fig. 10semi-log)
Correlation lengths are $\xi_1 = 1.08$, $\xi_2 = 1.13$ $\xi_3 = 1.12$, thus $\xi_1 \approx \xi_2 \approx \xi_3$.
Coefficients are $C_1 \approx C_2 \approx -C_3 > 0$.

Universality class of nTLL phase:PBC-TBC

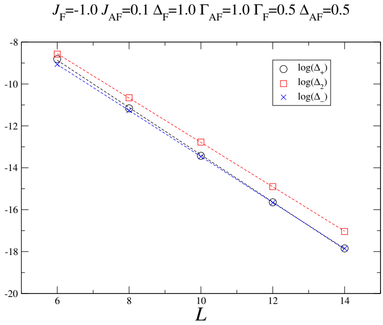
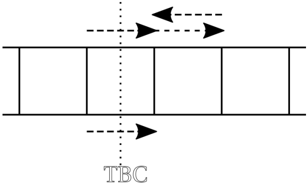
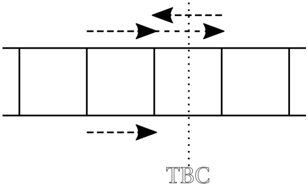


Figure: quasi degeneracy between PBC and TBC (Stripe Neel phase), $\Gamma_F = 0.5, \Delta_{AF} = 0.5$

Exponential quasi-degenaracy (Fig. 11semi-log)
Correlation lengths are $\xi_1 = 0.89$, $\xi_2 = 0.94$ $\xi_3 = 0.91$, thus $\xi_1 \approx \xi_2 \approx \xi_3$.
Coefficients are $C_1 \approx C_2 \approx -C_3 > 0$.

Universality class of nTLL phase:PBC-TBC (perturbation)





Phase-factor between PBC-TBC: $(-1)^2 = 1$

Summary

1. Takhtajan-Babujian
2. Ashkin-Teller multicritical point
   2.1 yTBC-zTBC method
   2.2 Now checking universality class etc.
3. Schulz multicritical point
   3.1 Both XY1-Haldane and nTLL(XY2)-Neel phase boundaries can be determined from the level crossing
   $M = 0, P = -1$ in TBC and $M = \pm 2, P = 0$ in PBC
   and they are BKT transition.
   One can also apply numerically these level cross for the S=1/2 spin ladder system without rung inversion symmetry.
   3.2 Universality class of the nTLL phase
      - c=1 CFT
      - quasi-degeneracy between TBC and PBC.

Future problem

- XY1-nTLL phase boundary
  level cross between
  - $\underline{M = 0, q = \pi \text{ in PBC and } M = \pm 1, q = \pi \text{ in PBC}}$
  - $\underline{M = 1 \text{ and } M = 2 \text{ in PBC}}$

  (tentative)
- Universality class of the Schulz multicritical point.
  c=3/2 CFT? not the SU(2) level 2 WZW.
- General spin S?